A Learning System of Quadratic Functions based on Scaffolding Support

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Abstract—In this study, we aimed to implement a learning system that supports learners to solve problems with respect to ‘Existence Domain of Solutions of Quadratic Functions (EDSQF)’. Our system also supports them to relearn previous units of quadratic functions. In addition, we designed a method for estimating learners’ weak points with their use histories on our system. In our system, learners can learn by using learning materials with respect to EDSQF and related units. Our system has a function that checks learners’ answers automatically and a function that provides explanations. Namely, learners can recognize immediately whether their answer is correct or not. Moreover, if they failed to solve problems, they can find explanations of those problems and challenge to solve related exercises. By providing these functions, learners can do self-learning with our system. Experimental results are proved that our system had a positive effect on learning EDSQF and related units to solve problems.

Keywords—Learning support system, Mathematics education, Quadratic functions, Scaffolding

I. INTRODUCTION

Mathematics is one of major subjects in school education. In recent years, achievement in mathematics of Japanese students is on the rise [1]. However, some students have consciousness such that they are not good at mathematics. Especially, some students are not good at ‘Quadratic Functions (QFs)’ [2]. QFs are dealt with as one of the high-school mathematics units. Students usually use and/or draw some graphs to tackle various problems of QFs. There is strong relationship among QFs, quadratic equations, and quadratic inequalities. Therefore, QFs are important to learn mathematics in Japan [3]. However, mathematics teachers think that it is difficult to teach QFs to students [4]. In addition, it is difficult for students to understand the relationship [2]. One of the units with respect to QFs is ‘Existence Domain of Solutions of QF (EDSQF)’. This is a unit that students need to use knowledge of QFs and/or quadratic inequalities to learn. It is taken up as an end-of-chapter problem of QFs in Japanese textbooks [5][6]. This research focuses on EDSQF.

Numerous support methods of mathematics learning have been proposed (e.g. [7][8][9][10][11][12]). Learning-support methods that focus on learning of QFs have been also proposed [9][11][12]. In these methods, however, it is assumed that learners have already mastered the previous units. Hence, if the learners have not mastered the previous units, those methods cannot support them well. For such learners, a certain support to learn from the previous units is needed to solve EDSQF problems.

In this study, we aimed to design and implement a learning system that supports learners to solve problems with respect to EDSQF. Our system also supports them to relearn previous units of QFs. Targets of our system are students who have already studied QFs. In addition, we aimed to design a method for estimating learners’ weak points with their use histories on our system. In our system, learners can learn by using learning materials with respect to EDSQF and related units. Our system has a function that checks learners’ answers automatically and a function that provides explanations. Namely, learners can recognize immediately whether their answer is correct or not. Moreover, if they failed to solve problems, they can find explanations of those problems and challenge to solve related exercises. By providing these functions, learners can do self-learning with our system.

The rest of this paper is structured as follows. In Section II, we introduce the related works and point out drawbacks of those works. Next, we design and implement our system in Section III. Then, in Section IV, we conduct a pilot experiment to investigate the effectiveness of our system. We discuss the results of our experiment in Section V and conclude this research in Section VI.

II. RELATED WORKS

In this section, we introduce SSDG [9], GSS [11], and GRAPES [12]. These are systems that support learners to study QFs.

In SSDG (Supporting System for Drawing Graphs), learners can study from definitions with respect to QFs to drawing some graphs. For support of self-learning, the system has a function that automatically checks whether a learner’s answer is correct or not (We call this function the ‘checking function’ from here). If an answer is incorrect on a problem, the system teaches and/or provides some explanations of the problem (We also call this system’s behavior the ‘explanation
function’ from here). By providing these functions, learners can study QFs as homework without teachers. However, SSDG does not support solving EDSQF problems.

In GSS, learners can tackle problems with respect to the maximum/minimum value of QFs by themselves. GSS has a checking function and an explanation function, and supports the case analysis. If a learner inputs some cases to GSS, the system analyzes each case and shows a corresponding graph. By providing various graphs, learners can observe those graphs and study the case analysis. However, GSS assumes that learners have already mastered the previous units such as QF graphs. If they do not master the previous units, it is difficult for them to use the system.

In GRAPES, learners can study graphs and locus of linear/quadratic functions that are taken up in high-school mathematics. They can observe various graphs and locus by operating with a mouse. The case analysis and the shift of graphs are available in the system. As for QFs, learners can set the domain of each variable of QFs. In that case, learners can increase and decrease each value of variables. By using GRAPES, learners can understand graphs to solve EDSQF problems. However, they cannot understand graph conditions necessary to solve them. In addition, it is difficult for beginners to use GRAPES because it has many operational functions.

III. SYSTEM OVERVIEW

A. Learning Contents

Figure 1 shows an example problem of EDSQF and its solution procedure. There are three important points in solving EDSQF problems [4][10]: (1) Learners draw a graph and observe the axis of the graph and the relation among the graph, x-axis, and y-axis. (2) Learners determine some conditions so that precondition(s) of an EDSQF problem are satisfied. (3) Learners make formulas so that those determined conditions are satisfied.

**Question:**
Find the existing domain of \( m \) when the quadratic function: \( y = x^2 - 2mx - m + 6 \) has two positive x-intercepts.

**Solution:**
At first, we draw a graph so that the precondition of given question is satisfied (see the figure).

Next, we determine some conditions so that the precondition is satisfied. The given formula can be deformed to
\[
y = (x - m)^2 - m^2 - m + 6
\]
Therefore, the graph is convex downward. The axis of graph is \( x = m \). The following three conditions (A) – (C) are needed so that the precondition of given question is satisfied:

(A) The given quadratic function has two x-intercepts.
\[
D = 4m^2 + 4m - 24 > 0
\]
\[
\therefore \quad m < -3, \quad m > 2
\]
(B) The axis of graph has a positive x-intercept.
\[
\therefore \quad x = m > 0
\]
(C) The axis of graph has a positive y-intercept.
\[
-m + 6 > 0
\]
\[
\therefore \quad m < 6
\]
According to (A) – (C), the existing domain of \( m \) is \( 2 < m < 6 \).

Fig. 1. Example Problem of EDSQF and Its Solution

![Fig. 1. Example Problem of EDSQF and Its Solution](image)

![Fig. 2. Relationship among ‘existence domain of solutions of QF’ and the related units](image)
The precondition of EDSQF problem shown in Figure 1 is ‘when the quadratic function: \( y = x^2 - 2mx - m + 6 \) has two positive \( x \)-intercepts’. In this research, we focus on the problems whose precondition is ‘when a quadratic function has two positive \( x \)-intercepts’ though there are some problems whose precondition is ‘when a quadratic function has two negative \( x \)-intercepts’ or ‘when a quadratic function has one positive \( x \)-intercept and one negative’ [5][6][13]. The reason is that problems including the precondition we focus on are taken up as introduction of EDSQF in textbooks [5][6].

To study EDSQF, learners need to acquire the following knowledge [14]: quadratic equations, linear functions, QFs, linear inequalities, quadratic inequalities, the relationship between real-number solutions and coefficients of quadratic equations, and the position relationship between a graph of QFs and the \( x \)-axis. Figure 2 shows a correlation diagram of these learning contents. Arrows shown in Figure 2 denote order relation of two units. That is, if a learner wants to understand a (current) unit, he/she needs to master the previous unit.

In this research, we design the system so that learners can solve EDSQF problems and master the previous unit shown in Figure 2 effectively.

B. System Design

To solve EDSQF problems, learners need to understand a solution procedure like Figure 1. However, if they do not master the previous units, it is difficult for them to find the solution procedure. In addition, they may forget some units even if they have already learned them. From these, we adopt a scaffolding approach [15]. That is, in our system, learners can study the units shown in Figure 2 step by step. We design some steps by referring a commercially-available website [2]. These steps consist of learning contents taken up in the period from junior-high schools to high schools. In our system, learners can study from linear functions. By studying from fundamental units, learners can be motivated. After studying linear functions, they can study in the order of quadratic equations, QFs, linear inequalities, the relationship between real-number solutions and coefficients of quadratic equations, the position relationship between a graph of QFs and the \( x \)-axis, quadratic inequalities, and EDSQF.

Here, we design necessary functions and forms to implement our system adopted the scaffolding approach. Firstly, we design the Question and Answer Form (QAF). A QAF consists of some fill-in-blank forms. By providing a QAF, learners can observe a solution procedure of a problem and answer the problem. Then, they can understand the solution procedure. Secondly, we introduce a checking function and an explanation function. Finally, we design a relearn function that supports learners to relearn the previous units. By providing this function, learners can study the previous units that they want to master.

On the other hand, we design an estimation method that can enumerate learners’ weak points on QFs and the related learning contents. The estimation method uses learners’ answer histories of each question. By introducing the estimation method, learners can understand their weak points and can study them intensively. Details are explained in Section III-D.

### Table I. Examples of Correspondence Relationship Between a Problem and Weak Points

<table>
<thead>
<tr>
<th>Problem</th>
<th>Weak Points</th>
</tr>
</thead>
<tbody>
<tr>
<td>To answer the ( y )-intercept and slope given by a linear function</td>
<td>Linear functions, Meaning of the ( x )- and ( y )-intercept of linear functions, Meaning of the slope of linear functions</td>
</tr>
<tr>
<td>To answer the axis of quadratic function: ( y = 2x^2 + 3 ) and the vertex of the parabola</td>
<td>Meaning of the axis and the vertex of a parabola (quadratic functions), Shift of graph of quadratic functions</td>
</tr>
<tr>
<td>To answer the solution of linear inequality: ( 2x - 7 &lt; 5x - 1 )</td>
<td>Sign (positive or negative) changing by transposition, ( \leq ) or ( \geq ) changing of inequalities</td>
</tr>
<tr>
<td>To answer the number of real-number solutions of quadratic equation: ( x^2 + 7x + 9 = 0 )</td>
<td>Discriminants, Relationship between the discriminant and the number of real-number solutions of a quadratic equation</td>
</tr>
<tr>
<td>To answer the number of ( x )-intercepts of quadratic function: ( y = x^2 + 5x + 3 )</td>
<td>Discriminants, Relationship between the discriminant and the number of ( x )-intercepts of a quadratic equation</td>
</tr>
<tr>
<td>To answer the existing domain of ( x ) given by quadratic inequality: ( (x - 2)(x - 4) &gt; 0 )</td>
<td>Property of products, Relationship between quadratic functions and quadratic inequalities</td>
</tr>
</tbody>
</table>

### C. Usage of Our System

The use flow of our system is as follows:

1. A learner answers a problem by using a QAF. After answering, our system checks the learner’s answers.
2. If all answers are correct, our system provides the next problem (and jump to (1)). If at least one answer is incorrect, our system makes the learner read and understand some explanations of the problem (and jump to (3)).
(3) The learner answers it again. If all answers are correct, our system provides the next question with respect to the previous unit (and jump to (1)). If at least one answer is incorrect, the learner understands the explanations and answers again.

Figure 3 shows an example of QAF. As shown in Figure 3, there are a few questions and its answer forms in a QAF. If a learner presses the ‘Check’ button, our system checks whether the answers are correct or not. If a learner’s answer is incorrect, his/her answer is highlighted. Then, our system provides some buttons to show some explanations of the problem. Those explanations show in a different window. If all of a learner’s answers are correct, the ‘Next problem’ button is available. As for answer forms, learners can input numbers 0-9 and a minus symbol only.

If a learner’s answer is incorrect for the first checking and the answer is correct in checking from the second time, our system provides problem(s) with respect to the previous units. For example, the previous units of a problem including such a quadratic function: ‘\( y = ax^2 + bx + c \)’ are linear functions and completing the square of quadratic equations. If a learner does not solve the problem, our system provides two buttons to relearn linear functions and completing the square. Learners can decide whether they relearn them or not.

D. Design of Estimation Method for Enumerating Learners’ Weak Points

In this section, we explain an estimation method for enumerating learners’ weak points. Table I is the correspondence relationship table between a problem and weak points and is used to enumerate learners’ weak points. For example, a learner does not solve a problem with respect to ‘To understand a learning content, but estimation failed. Learners cannot understand a learning content’.

IV. EVALUATION EXPERIMENT

To investigate the effectiveness of our system, we conducted an evaluation experiment with 12 undergraduate/graduate students (including 2 international students) who have already learned QFs. We investigated whether participants who have some weak points with respect to QFs overcame them or not by using our system.

A. Methods

In the experiment, all participants were required to take a pre-test for 30 minutes. In the pre-test, we prepared 14 questions, which consisted of two questions of linear equations, two of the axis of QFs, one of linear equations, one of the number of real-number solutions of quadratic equations, one of the number of \( x \)-intercepts of QFs, four of quadratic equations, one of simultaneous inequalities, one of QFs including a parameter, one of EDSQF. After the pre-test, all participants were required to study EDSQF and the related learning contents by using our system for 40 minutes. After the studying session, we conducted a post-test for 30 minutes and an interview. The post-test was similar to the pre-test. Then, we investigate the effectiveness of our system by using the pre- and post-test results.

Full marks of each test were 30. 5 of 30 are full marks of the EDSQF problem. As for this, we checked whether each participant solved the following five sub-questions or not:

(1) Whether the discriminant is correct or not.
(2) Whether the axis of given QF is correct or not.
(3) Whether the \( y \)-intercept of given QF is correct or not.
(4) Whether the intersection of (1) – (3) is correct or not.
(5) Whether the EDSQF is correct or not.

One point was assigned to each sub-question.

<table>
<thead>
<tr>
<th>Participant</th>
<th>Total Score</th>
<th>Score of EDSQF</th>
<th>Total Score</th>
<th>Score of EDSQF</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>28</td>
<td>5</td>
<td>28</td>
<td>5</td>
</tr>
<tr>
<td>B</td>
<td>26</td>
<td>2</td>
<td>30</td>
<td>5</td>
</tr>
<tr>
<td>C</td>
<td>25</td>
<td>1</td>
<td>27</td>
<td>2</td>
</tr>
<tr>
<td>D</td>
<td>23</td>
<td>1</td>
<td>30</td>
<td>5</td>
</tr>
<tr>
<td>E</td>
<td>23</td>
<td>2</td>
<td>30</td>
<td>5</td>
</tr>
<tr>
<td>F</td>
<td>23</td>
<td>1</td>
<td>25</td>
<td>2</td>
</tr>
<tr>
<td>G</td>
<td>23</td>
<td>1</td>
<td>27</td>
<td>2</td>
</tr>
<tr>
<td>H</td>
<td>21</td>
<td>1</td>
<td>27</td>
<td>4</td>
</tr>
<tr>
<td>I</td>
<td>19</td>
<td>2</td>
<td>29</td>
<td>5</td>
</tr>
<tr>
<td>J</td>
<td>11</td>
<td>0</td>
<td>17</td>
<td>1</td>
</tr>
<tr>
<td>K</td>
<td>10</td>
<td>0</td>
<td>25</td>
<td>2</td>
</tr>
<tr>
<td>L</td>
<td>10</td>
<td>0</td>
<td>25</td>
<td>1</td>
</tr>
</tbody>
</table>

TABLE II. RESULTS OF PRE- AND POST-TESTS

<table>
<thead>
<tr>
<th>Grade</th>
<th>Pre-test</th>
<th>Estimation</th>
<th>Post-test</th>
<th>Comment</th>
</tr>
</thead>
<tbody>
<tr>
<td>A1</td>
<td>*</td>
<td>NW</td>
<td>*</td>
<td>Estimation succeeded. Learners can understand a learning content</td>
</tr>
<tr>
<td>A2</td>
<td></td>
<td>W</td>
<td></td>
<td>Estimation succeeded. Learners cannot understand a learning content</td>
</tr>
<tr>
<td>B1</td>
<td>NW</td>
<td></td>
<td></td>
<td>Probably estimation succeeded. Learners can understand a learning content by using our system.</td>
</tr>
<tr>
<td>B2</td>
<td>W</td>
<td>*</td>
<td></td>
<td>Probably estimation succeeded. Learners can understand a learning content at the time of post-test</td>
</tr>
<tr>
<td>C1</td>
<td>*</td>
<td>W</td>
<td>*</td>
<td>Estimation failed. Learners can understand a learning content, but our system judged that it is a weak point.</td>
</tr>
<tr>
<td>C2</td>
<td>NW</td>
<td></td>
<td></td>
<td>Estimation failed. Learners cannot understand a learning content, but our system judged that it is not a weak point.</td>
</tr>
</tbody>
</table>

*: correct, NW: not weak point, W: weak point
In the study session, all participants are required to fill in blanks of a particular paper to record their use histories of our system. As for validity investigation of our estimation method, after the study session, we enumerated each participant’s weak points by using his/her answer histories on our system. Then, we compared those weak points, the pre-test results, and the post-test results. In addition, in each learner and learning content, we estimated whether a learning content was a participant’s weak point or not.

### B. Results of Pre- and Post-Tests

Table II shows the results of pre- and post-tests. Most of participants’ post-test scores were higher than pre-test scores. A 2-paired t-test revealed that a significant difference between mean scores of pre- and post-tests was noted (t(11) = 4.70, p < 0.01). From this, we can suppose that the post-test scores were statistically higher, and use of our system had a positive effect. In addition, A 2-paired t-test revealed that a significant difference between mean scores of the EDSQF problem given in pre- and post-tests was noted (t(11) = 5.35, p < 0.01). From this, we can also suppose that there is a positive effect in learning EDSQF by using our system.

### C. Results of Validity Investigation of Weak Points Estimation Method

After comparing learners’ weak points, pre-test results, and post-test results, we found six patterns of relationship among them. Those patterns are shown in Table III. We determined the estimation grades called A1, A2, B1, B2, C1, and C2 based on those patterns. For example, for a participant’s weak point (a learning content), when the participant was not able to solve a problem including the learning content in both pre- and post-tests, the estimation grade was A2. In such case, the system can estimate his/her weak point. On the other hand, for a participant’s weak point, when the participant was able to solve one question in both pre- and post-tests, the estimation grade was C1. In such case, our system’s estimation failed. Thus, we analyzed each weak point.

Table IV shows the analysis results. Total cases were 144. Correct rate of estimation was 88% in total. In addition, correct rate in weak points’ estimation (A2, B1, and C1) was 29%. From these, our estimation method was almost acceptable. However, we considered that estimation probably succeeded in match the result of pre- or post-test.

### V. DISCUSSION

The experimental results revealed that learning EDSQF by using our system had a positive effect. In this section, we discuss the experimental results.

#### A. Learning by using Our System

Although 3 participants had studied all problems in our system, they were not able to solve an EDSQF problem in the post-test. One participant made a careless mistake such as writing a symbol ‘m’ instead of ‘x’. The others overlooked the condition with respect to the axis of given graph. As for this, participants had to study each EDSQF problem over a span of 5 QAFs in our system. Those participants may not be able to acquire knowledge of the axis of a given graph because the amount of blanks to be filled in was little in the QAF with respect to the axis of a given graph. Improvement of our system is needed so that any learner pays attention to all learning contents.

#### B. Validity of Our Estimation Method

According to Table IV, 3 cases and 2 cases are existed in C1 of ‘Factorization’ and C1 of ‘Completing the square’ respectively. Problems with respect to these learning contents were provided the early stage of our system. Participants who have the weak point estimated as the C1 case have already understood ‘Factorization’ or ‘Completing the square’, but they might not solve problems in our system because they were not used to our system. Therefore, our estimation failed. As for
other C1 cases, our estimation might fail because the usability of our system was low. In the interview after conduction the post-test, some participants responded that our system’s interfaces were not good. Hence, some participants might not study learning contents well and solve problems in our system.

Except for A1 cases, B2 is the most cases in total. One of the reasons is knowledge of common mathematical formula. Participants who have the weak point estimated as the B2 case had already understood some common mathematical formulas, but they might not make use of them in the pre-test. Another reason is careless mistakes. Some participants had already understood a learning content, but they were not able to solve a problem including the learning content in our system because of their careless mistake. However, we can assume that some participants have understood a learning content by using our system. Therefore, they were able to solve a problem including the learning content in the post-test though they were not able to do in the pre-test.

6 cases were estimated as the C2 case. The relationship between quadratic functions and quadratic inequalities is necessary knowledge to solve questions including quadratic inequalities. Participants can see some hints and/or some graphs in solving a problem of quadratic inequalities by using our system. Therefore, those participants can solve the problem. However, those participants need to imagine or draw a graph as well as the position relationship among the graph, $x$-axis, and $y$-axis in pre- and post-test. Some participants who had the weak point estimated as the C2 case were not able to make a graph that satisfies preconditions of given problems. From this, those participants were able to solve questions in our system because our system provided some hints.

VI. CONCLUSIONS AND FUTURE WORKS

In this research, we have designed and implemented a system that supports learners to study ‘Existence Domain of Solutions of Quadratic Functions (EDSQF)’ and the related learning contents which are the previous units of EDSQF. Our system has a function that automatically checks whether a learner’s answer is correct or not, a function that provides some explanations with respect to a problem which a learner cannot solve, and a function that provides problem(s) with respect to the previous units. In addition, we designed an estimation method that can enumerate a learner’s weak points with respect to solving EDSQF.

To measure the effectiveness of our system, we conducted a pilot experiment with 12 undergraduate or graduate students. After a pre-test, all participants studied EDSQF and the related learning contents in our system. After the study session, a post-test was carried out. As for scores of EDSQF problem, a statistical analysis revealed that learning by using our system had a positive effect.

On the other hand, we estimated each learner’s weak points by using their answer histories on our system. Success rate of our estimation was 88%. Success rate of our estimation in case our system estimated that a learner has a weak point with respect to a learning content was 29%. In addition, failure rate of our estimation in case our system estimated that learner does not have any weak point in a learning content was 4%.

Our future work is improvement of our system’s interfaces. Improvement of our estimation method and conducting an experiment with junior-high school students are also our future works.

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