Relations among Weyl modules, Demazure modules and finite crystals

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Notation

∘ $\mathfrak{g}$: simple Lie algebra with index set $I = \{1, \dots, n\}$, ∘ $\mathfrak{h}$: Cartan subalgebra,
∘ $\alpha_1, \dots, \alpha_n$: simple roots, ∘ $\varpi_1, \dots, \varpi_n$: fundamental weights, ∘ W: Weyl group,
∘ $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$: triangular decomposition, ∘ $P = \bigoplus_i \mathbb{Z}\varpi_i$, $P_+ = \bigoplus_i \mathbb{Z}_{\geq 0}\varpi_i$,
∘ $\{e_i, h_i, f_i \mid i \in I\}$: Chevalley generators,
∘ $\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}d$: the affine Lie algebra associated to $\mathfrak{g}$,
∘ $\Lambda_0, \dots, \Lambda_n$: fundamental weights of $\widehat{\mathfrak{g}}$, ∘ δ: null root of $\widehat{\mathfrak{g}}$.

Definitions

• Weyl module $W(\Lambda)$

The Lie subalgebra $\mathfrak{g} \otimes \mathbb{C}[t] \oplus \mathbb{C}d \subseteq \widehat{\mathfrak{g}}$ is called the current algebra.

Definition. *For a dominant integral weight $\lambda \in P_+$, the Weyl module $W(\lambda)$ is a $\mathfrak{g} \otimes \mathbb{C}[t] \oplus \mathbb{C}d$-module generated by an element v with the relations:*

$$\mathfrak{n}_+ \otimes \mathbb{C}[t].v = 0, \; h \otimes t^s.v = \delta_{s0}\langle \lambda, h\rangle v \text{ for } h \in \mathfrak{h}, s \in \mathbb{Z}_{\geq 0},$$
$$f_i^{\langle \lambda, h_i\rangle + 1}.v = 0 \text{ for } i \in I, \text{ and } d.v = 0.$$

We denote the $\mathbb{Z}$-graded character of $W(\lambda)$ by

$$\operatorname{ch} W(\lambda) = \sum_{\mu \in P, m \in \mathbb{Z}} q^m e(\mu) \dim W(\lambda)_{\mu + m\delta}.$$

• Demazure module $D(\lambda, m)$ and Demazure crystal $B(\lambda, m)$

For $\lambda \in P_+$ and $m \in \mathbb{Z}$, let Λ be the unique dominant integral weight of $\widehat{\mathfrak{g}}$ such that

$$w(\Lambda) = \lambda + \Lambda_0 + m\delta \quad \text{for some element } w \text{ of the affine Weyl group.}$$

Definition. *The Demazure module $D(\lambda, m)$ is the $\mathfrak{g} \otimes \mathbb{C}[t] \oplus \mathbb{C}d$-submodule of the irreducible highest weight $\widehat{\mathfrak{g}}$-module $V(\Lambda)$ generated by the 1-dimensional weight space $V(\Lambda)_{\lambda + \Lambda_0 + m\delta}$.*

It is well-known that the q-analog of $V(\Lambda)$ has a crystal basis $B(\Lambda)$, and it is also known that the q-analog of $D(\lambda, m)$ also has a crystal basis ([K1]), which is denoted by $B(\lambda, m)$ and called the *Demazure crystal*.

• Crystal basis of a fundamental representation and enegy function

For $i \in I$, let $W_q(\varpi_i)$ denote the *fundamental representation* of the quantum affine algebra $U_q'(\widehat{\mathfrak{g}})$ defined by Kashiwara (cf. [K2]), and B_{ϖ_i} its crystal basis. When $\lambda = \sum_i \lambda_i \varpi_i$, we denote by B_λ the tensor product

$$B_\lambda = \bigotimes_i B_{\varpi_i}^{\otimes \lambda_i}.$$

The *enegy function* $E_\lambda : B_\lambda \to \mathbb{Z}$ is a certain $\mathbb{Z}$-function defined in a combinatorial way (since its definition is a little complicated, we omit it here. See [HKOTY], for example). We define a $\mathbb{Z}$-grading on B_λ via the function $-E_\lambda$.

Remark. *It is known that the crystal B_λ and the energy function E_λ can be realized using the Lakshmibai-Seshadri paths* ([NS]). *To prove the main theorem below, these realizations are essentially used.*

Main Theorem

Theorem ([Na]). (1) *$W(\lambda)$ has a filtration $0 = W_0 \subseteq W_1 \subseteq \cdots \subseteq W_k = W(\lambda)$ such that each quotient W_j/W_{j+1} is isomorphic to some Demazure module $D(\mu_j, m_j)$ for $1 \leq j \leq k$.*

(2) *The subset $u_{\Lambda_0} \otimes B_\lambda$ of the crystal $B(\Lambda_0) \otimes B_\lambda$ (u_{Λ_0} is the highest weight element of $B(\Lambda_0)$) is isomorphic to the disjoint union of the Demazure crystals $\coprod_{1 \leq j \leq k} B(\mu_j, m_j)$, where μ_j and m_j are the ones in* (1). *Moreover, this isomorphism preserves their $\mathbb{Z}$-gradings.*

Remark. *If $\mathfrak{g}$ is of type ADE, then $k = 1$ and the Weyl module $W(\lambda)$ is in fact isomorphic to the Demazure module $D(\lambda, 0)$. This fact was previously proved by Fourier and Littelmann* [FL].

$X = M$ conjecture

Definition. *Let $\lambda, \mu \in P_+$.*

(1) *The 1-dimensional sum $X(B_\lambda, \mu, q)$ is defined by*

$$X(B_\lambda, \mu, q) = \sum_{\substack{b \in B_\lambda \\ \tilde{e}_j b = 0 \ (j \in I) \\ \mathrm{wt}(b) = \mu}} q^{E_\lambda(b)}.$$

(2) *The fermionic form $M(\lambda, \mu, q)$ is defined as follows:*

$$M(\lambda, \mu, q) = \sum_{\mathbf{m} \in S} q^{c_{\mathbf{m}}} \prod_{i \in I, k \geq 1} \begin{bmatrix} p^{\mathbf{m}}_{k,i} + m_k^{(i)} \\ m_k^{(i)} \end{bmatrix}_q,$$

where

$$S = \left\{ \mathbf{m} = (m_k^{(i)})_{\substack{i \in I \\ k \geq 1}} \;\middle|\; \sum_{i \in I, k \geq 1} k m_k^{(i)} \alpha_i = \lambda - \mu \right\}, \; p^{\mathbf{m}}_{k,i} = \langle \lambda, h_i \rangle - \sum_{j \in I} (\alpha_i, \alpha_j) \sum_{\ell \geq 1} \min\{k, \ell\} m_\ell^{(j)},$$
$$c_{\mathbf{m}} = 2^{-1} \sum_{i,j \in I} (\alpha_i, \alpha_j) \sum_{k, \ell \geq 1} \min\{k, \ell\} m_k^{(i)} m_\ell^{(j)} - \sum_{i \in I, k \geq 1} \langle \lambda, h_i \rangle m_k^{(i)}.$$

By our main theorem, we have

$$\operatorname{ch} W(\lambda) = \sum_{1 \leq j \leq k} \operatorname{ch} D(\mu_j, m_j) = \sum_{1 \leq j \leq k} \sum_{b \in B(\mu_j, m_j)} q^{\langle d, \mathrm{wt}(b)\rangle} e\big(\mathrm{wt}_P(b)\big)$$
$$= \sum_{b \in B_\lambda} q^{-E_\lambda(b)} e\big(\mathrm{wt}(b)\big) = \sum_{\mu \in P_+} X(B_\lambda, \mu, q^{-1}) \operatorname{ch} V_{\mathfrak{g}}(\mu),$$

where $V_{\mathfrak{g}}(\mu)$ denotes the irreducible $\mathfrak{g}$-module. On the other hand, the following theorem was proved by Di Francesco and Kedem:

Theorem ([DFK]).

$$\operatorname{ch} W(\lambda) = \sum_{\mu \in P_+} M(\lambda, \mu, q^{-1}) \operatorname{ch} V_{\mathfrak{g}}(\mu).$$

Hence, we have the following corollary, which is a special case of the $X = M$ conjecture (cf. [HKOTY]):

Corollary. *We have*

$$X(B_\lambda, \mu, q) = M(\lambda, \mu, q).$$

Remark. *The general $X = M$ conjecture is formulated on more general crystals called the Kirillov-Reshetikhin crystals.*

References

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