Finite-dimensional modules over a quantum loop algebra and their classical limits

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Plan

1. Definition of a quantum loop algebra $U_q(\mathcal{L}\mathfrak{g})$ (QLA)

   - motivation, basic properties

2. fin. dim. $U_q(\mathfrak{g})$-modules

3. fin. dim. $U_q(\mathcal{L}\mathfrak{g})$-modules

4. Results

   (1) classical limits of simple modules

   (2) classical limits of tensor products

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Definition of a quantum loop algebra $U_q(\mathcal{L}\mathfrak{g})$

$\mathfrak{g}$: simple Lie algebra/$\mathbb{C}$

$\big($e.g. $\mathfrak{sl}_n := \{X \in \mathrm{Mat}(n,\mathbb{C}) \mid \mathrm{tr}\, X = 0\}$, $[X,Y] = XY - YX\big)$

$\mathcal{L}\mathfrak{g} := \mathfrak{g} \otimes \mathbb{C}[t,t^{-1}]$: loop algebra $([X \otimes f, Y \otimes g] = [X,Y] \otimes fg)$

Quantum loop algebra $U_q(\mathcal{L}\mathfrak{g})$ (QLA)

$=$ the quantized enveloping algebra assoc. with $\mathcal{L}\mathfrak{g}$.

$\mathfrak{a}$ (Lie algebra such as $\mathfrak{g}$, $\mathcal{L}\mathfrak{g}$)

$\Longrightarrow$ $U(\mathfrak{a})$: universal enveloping algebra of $\mathfrak{a}$ (associative $\mathbb{C}$-algebra)

$\xRightarrow{q\text{-deform}}$ $U_q(\mathfrak{a})$: the quantized env. algebra (associative $\mathbb{C}(q)$-algebra)

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First step $\mathfrak{a}$: Lie algebra $\Rightarrow$ $U(\mathfrak{a})$: universal enveloping algebra

$U(\mathfrak{a}) := T(\mathfrak{a})/\langle XY - YX - [X,Y] \mid X, Y \in \mathfrak{a}\rangle$

($T(\mathfrak{a}) := \bigoplus_{k=0}^{\infty} T^k(\mathfrak{a})$: tensor algebra of $\mathfrak{a}$,

i.e. the free (associative) algebra generated by $\mathfrak{a}$)

Rem. V is an $\mathfrak{a}$-module $\Leftrightarrow$ V is a $U(\mathfrak{a})$-module

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Second step $U(\mathfrak{a})$: univ. env. alg. $\overset{q-\text{defom}}{\Longrightarrow}$ $U_q(\mathfrak{a})$: quantized env. alg.

$U_q(\mathfrak{a})$ is defined as a q-deformation of $U(\mathfrak{a})$ i.e. "$\lim_{q\to 1} U_q(\mathfrak{a}) = U(\mathfrak{a})$".

Ex.1 $\mathfrak{a} = \mathfrak{sl}_2 = \mathbb{C}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \oplus \mathbb{C}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \oplus \mathbb{C}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \mathbb{C}e \oplus \mathbb{C}h \oplus \mathbb{C}f$

$[e, f] = h, \quad [h, e] = 2e, \quad [h, f] = -2f$

$U(\mathfrak{sl}_2)$: the $\mathbb{C}$-algebra generated by e, h, f with relations:

$ef - fe = h, \quad he - eh = 2e, \quad hf - fh = -2f \quad - (*)$

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∘ e, h, f satisfy the rel. in $U(\mathfrak{sl}_2)$, ∘ $\tilde{e}, h, \tilde{f}$ satisfy the rel. in $U(\mathfrak{sl}_2)$,

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Q. There are any number of algebras which specialize to $U(\mathfrak{a})$ at $q = 1$. Why the q-derom. $U_q(\mathfrak{a})$ is particulaly important?

Note $U(\mathfrak{a})$ is a Hopf algebra!

i.e. $^\exists$coproduct $\Delta\colon U(\mathfrak{a}) \to U(\mathfrak{a}) \otimes U(\mathfrak{a})$, $^\exists$counit $\varepsilon\colon U(\mathfrak{a}) \to \mathbb{C}$, $^\exists$antipode $S\colon U(\mathfrak{a}) \xrightarrow{\sim} U(\mathfrak{a})^{\mathrm{op}}$ with some compatibility.

($\Rightarrow$ If V, W are $U(\mathfrak{a})$-mod., then $V \otimes W$, $\mathbb{C}$, V^* become $U(\mathfrak{a})$-modules.)

Theorem

$U_q(\mathfrak{a})$ has a Hopf algebra structure.

That is, $U_q(\mathfrak{a})$ is q-deform. of $U(\mathfrak{a})$ as a Hopf algebra!

Rem. Moreover, $U_q(\mathfrak{a})$ is a quasi-triangular Hopf alg., which implies that a solution (R-matrix) of the Yang-Baxter equation can be obtained from a pair of $U_q(\mathfrak{a})$-modules.

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Motivation to the study of fin. dim. $U_q(\mathcal{L}\mathfrak{g})$-mod.

1. All the $U_q(\mathcal{L}\mathfrak{g})$-modules are too large to controll.
2. f.d. $U_q(\mathcal{L}\mathfrak{g})$-mod. have rich structures:
   - The category of f.d. $U_q(\mathcal{L}\mathfrak{g})$-mod. is a monoidal category via $\otimes$
     (Recall $U_q(\mathcal{L}\mathfrak{g})$: Hopf alg. $\Rightarrow V \otimes W$: $U_q(\mathcal{L}\mathfrak{g})$-mod.)
   - not semisimple ◦ uncountable simple modules
3. have connection to other fields:
   - integrable system
   - geometry on quiver variety (instanton moduli on ALE space)
   - combinatorics, cluster algebra, . . . , etc.
4. many problems remain unsolved (str. of simple mod., tensor prod.)
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Motivation to the study of fin. dim. $U_q(\mathcal{L}\mathfrak{g})$-mod.

1. All the $U_q(\mathcal{L}\mathfrak{g})$-modules are too large to controll.
2. f.d. $U_q(\mathcal{L}\mathfrak{g})$-mod. have rich structures:
   - ◦ The category of f.d. $U_q(\mathcal{L}\mathfrak{g})$-mod. is a monoidal category via $\otimes$

     (Recall $U_q(\mathcal{L}\mathfrak{g})$: Hopf alg. $\Rightarrow V \otimes W$: $U_q(\mathcal{L}\mathfrak{g})$-mod.)
   - ◦ not semisimple ◦ uncountable simple modules
3. have connection to other fields:
   - ◦ integrable system
   - ◦ geometry on quiver variety (instanton moduli on ALE space)
   - ◦ combinatorics, cluster algebra, . . . , etc.
4. many problems remain unsolved (str. of simple mod., tensor prod.)

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fin. dim. $U_q(\mathfrak{g})$-modules

Proposition

1. f.d. $U_q(\mathfrak{g})$-mod. are semisimple (i.e. ${}^\forall V \cong \bigoplus(\text{simple})$)
2. {f.d. simple $U_q(\mathfrak{g})$-mod.} $\overset{\text{1-1}}{\leftrightarrow}$ $\mathbb{Z}^n_{\geq 0}$

$$V_q(\lambda) \leftrightarrow \lambda = (\lambda_1, \ldots, \lambda_n)$$

Q. ◦ Study the structure of simple mod. $V_q(\lambda)$, such as character $\operatorname{ch} V_q(\lambda)$.

($V = \bigoplus_\alpha V_\alpha$: simultaneous eigensp. dec. w.r.t. a comm. subalg.

$$\Rightarrow \operatorname{ch} V := \textstyle\sum_\alpha t^\alpha \dim V_\alpha \in \mathbb{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}])$$

◦ Given two simple $U_q(\mathfrak{g})$-mod. $V_q(\lambda)$, $V_q(\mu)$, determine the multiplicities $[V_q(\lambda) \otimes V_q(\mu) : V_q(\nu)]$ for every ν.

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Definition (Classical limit)

V: f.d. $U_q(\mathfrak{g})$-mod. $\overset{\lim_{q\to 1}}{\Rightarrow}$ $\overline{V}$: f.d. $U(\mathfrak{g})$-mod. (=$\mathfrak{g}$-mod.)

(Recall "$\lim_{q\to 1} U_q(\mathfrak{g}) = U(\mathfrak{g})$")

Proposition

1. For every $\lambda \in \mathbb{Z}^n_{\geq 0}$, $V(\lambda) := \overline{V_q(\lambda)}$ is a simple $\mathfrak{g}$-mod., and $V_q(\lambda) \mapsto V(\lambda)$ induces a bij. from {f.d. simple $U_q(\mathfrak{g})$-mod.} to {f.d. simple $\mathfrak{g}$-mod.}

   ($\Rightarrow \operatorname{ch} V_q(\lambda) = \operatorname{ch} V(\lambda)$ ← Weyl's character formula)

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fin. dim. $U_q(\mathcal{L}\mathfrak{g})$-modules

Theorem (Chari-Pressley, 95)

$\{$f.d. simple $U_q(\mathcal{L}\mathfrak{g})$-mod.$\}$

$$\overset{1\text{-}1}{\leftrightarrow} \{\boldsymbol{\pi} = (\pi_1(u), \ldots, \pi_n(u)) \mid \pi_k(u) \in 1 + u\mathbb{C}(q)[u]\}$$

Denote the simples by $L_q(\boldsymbol{\pi})$ ($\boldsymbol{\pi}$ are called the Drinfeld polynomials).

Rem. f.d. $U_q(\mathcal{L}\mathfrak{g})$-mod. are not semisimple.

Q. (i) Study the structure of $L_q(\boldsymbol{\pi})$.

(ii) Study $L_q(\boldsymbol{\pi}) \otimes L_q(\boldsymbol{\pi}')$.

Rem. By the geometric rep. theory on quiver varieties, H. Nakajima has obtained an algorithm to give characters (or q-characters) of every $L_q(\boldsymbol{\pi})$. The algorithm is very complicated, however, and a closed formula is still to be established (at least to some "good class" of simple modules).

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V: f.d. $U_q(\mathcal{L}\mathfrak{g})$-mod. wih some conditions (e.g. almost all simples)

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Goal Study the structure of V by observing $\overline{V}$ instead.

good point ∘ $U(\mathcal{L}\mathfrak{g})$ is a much simpler alg. than $U_q(\mathcal{L}\mathfrak{g})$.

∘ $\mathfrak{g} = \mathfrak{g} \otimes 1 \subseteq \mathfrak{g} \otimes \mathbb{C}[t^{\pm 1}] = \mathcal{L}\mathfrak{g}$: Lie subalg., $U_q(\mathfrak{g}) \subseteq U_q(\mathcal{L}\mathfrak{g})$: subalg., and $V \cong_{U_q(\mathfrak{g})} \bigoplus_\lambda V_q(\lambda)^{\oplus m_\lambda} \Leftrightarrow \overline{V} \cong_{\mathfrak{g}} \bigoplus_\lambda V(\lambda)^{\oplus m_\lambda}$, that is, taking the classical limit does not loose any information of the $U_q(\mathfrak{g})$-module structure (in particular, $\mathrm{ch}\, V = \mathrm{ch}\, \overline{V}$)

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f.d. $\mathcal{L}\mathfrak{g}$-modules are not semisimple, and even V is simple, $\overline{V}$ is rarely simple $\Rightarrow$ need to treat a nonsimple, indec. $\mathcal{L}\mathfrak{g}$-mod.

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(1) classical limits of simple modules

Strategy

Given V: simple $U_q(\mathcal{L}\mathfrak{g})$-mod. $\overset{\lim_{q\to 1}}{\Rightarrow}$ $\overline{V}$: $U(\mathcal{L}\mathfrak{g})$-mod.($=\mathcal{L}\mathfrak{g}$-mod.)

Basically, the study of $\overline{V}$ is divided into the following two steps:

1. $\overline{V}$ is generated by a distinguished vector called an ℓ-highest weight vector (ℓ-h.w.v.) v. The first step is to find a "good" defining relations of v, i.e., find a "good" subset $S \subset U(\mathcal{L}\mathfrak{g})$ s.t. $\mathrm{Ann}(v) = U(\mathcal{L}\mathfrak{g})\, S$ $\;(\Leftrightarrow U(\mathcal{L}\mathfrak{g})/U(\mathcal{L}\mathfrak{g})\, S \overset{\sim}{\to} \overline{V})$.
2. Using $U(\mathcal{L}\mathfrak{g})/U(\mathcal{L}\mathfrak{g})\, S \cong \overline{V}$, determine its character, simple $\mathfrak{g}$-module dec., etc.

As a consequence, we obtain the information on the structure of V such as character, simple $U_q(\mathfrak{g})$-mod. dec., etc.

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Notation and Definition

Recall

$\{\text{f.d. simple } U_q(\mathcal{L}\mathfrak{g})\text{-mod.}\} \overset{\text{1-1}}{\leftrightarrow} \{(\pi_1(u), \dots, \pi_n(u)) \mid \pi_k(0) = 1\} =: \mathcal{D}$

For $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n_{\geq 0}$, $\mathcal{D}^\lambda := \{(\pi_1, \dots, \pi_n) \mid \deg \pi_k = \lambda_k\} \subseteq \mathcal{D}$

Fact If $\boldsymbol{\pi} \in \mathcal{D}^\lambda$, $L_q(\boldsymbol{\pi}) \cong_{U_q(\mathfrak{g})} V_q(\lambda) \oplus$ (smaller simple $U_q(\mathfrak{g})$-mod.).

Definition

For $\lambda \in \mathbb{Z}^n_{\geq 0}$, $L_q(\boldsymbol{\pi})$ is a minimal affinization with weight λ

$\overset{\text{def}}{\Leftrightarrow}$ smallest dimensional among $\{L_q(\boldsymbol{\pi}) \mid \boldsymbol{\pi} \in \mathcal{D}^\lambda\}$.

The case $\lambda = \ell e_i = (0, \dots, 0, \ell, 0, \dots, 0)$ $(1 \leq i \leq n, \ell \in \mathbb{Z}_{>0})$ is particulary important in integrable system. These special min. affiniz. are called Kirillov-Reshetikhin (KR) modules with weight ℓe_i

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classical limits of Kirillov-Reshetikhin module

Let $\{W^{i,\ell}(a) \mid a \in \mathbb{C}(q)^\times\}$ denote the KR modules with weight ℓe_i.

Study of KR mod. in integrable system

$\rightsquigarrow$ conjectural formula for simple $U_q(\mathfrak{g})$-mod. dec.:

$W^{i,\ell}(a) \cong_{U_q(\mathfrak{g})} \bigoplus V_q(\lambda)^{\oplus m_{i,\ell}(\lambda)}$ (KR conjectrue)

Theorem (Chari, 01)

Assume that $\mathfrak{g}$ is of type A_n, B_n, C_n or D_n, and $v \in W^{i,\ell}(a)$ is ℓ-h.w.v.

(i) $\mathrm{Ann}(v) = U(\mathcal{L}\mathfrak{g})\, S$ with a certain subset S of elements in $U(\mathcal{L}\mathfrak{g})$

$$\left(S = \mathfrak{n}_+[t] \cup (\mathfrak{h} \otimes (t - a(1))) \cup \{f_i^{\ell+1}, f_j\ (j \neq i), f_i \otimes t\}\right)$$

(ii) KR conj. holds.

Recall Simple Lie algebras are classified into types

A_n ($\mathfrak{sl}_{n+1}$), B_n ($\mathfrak{so}_{2n+1}$), C_n ($\mathfrak{sp}_{2n}$), D_n ($\mathfrak{so}_{2n}$), E_6, E_7, E_8, F_4, G_2.

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sketch of proof

Fact If $\mathfrak{g}$ is of type $ABCD$, $m_{i,\ell}(\lambda) \leq 1$ for any λ.

- $U(\mathcal{L}\mathfrak{g})/U(\mathcal{L}\mathfrak{g})\,S \twoheadrightarrow \overline{V}$ is easy to prove.
- Prove $[U(\mathcal{L}\mathfrak{g})/U(\mathcal{L}\mathfrak{g})\,S : V(\lambda)] \leq m_{i,\ell}(\lambda)$.
- Prove $[W^{i,\ell}(a) : V_q(\lambda)] \geq m_{i,\ell}(\lambda)$ (hard calculation in $U_q(\mathcal{L}\mathfrak{g})$) $\square$

It seems difficult to apply this proof to wider class of simple modules!

Rem. KR conj. has been proved in general types using the theory of q-characters.

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class. lim. of minimal affinizations

Theorem (N, Li-N, '13-'15)

Assume $\mathfrak{g}$ is of type $ABCD$ or G_2, and let v be an ℓ-h.w.v. of a minimal affinization W with weight $\lambda \in \mathbb{Z}^n_{\geq 0}$.

Then there exists a "good" subset $S \subseteq U(\mathcal{L}\mathfrak{g})$ s.t. $\mathrm{Ann}(v) = U(\mathcal{L}\mathfrak{g})\,S$.

Corollary

By analizing $U(\mathcal{L}\mathfrak{g})/U(\mathcal{L}\mathfrak{g})\,S$, we obtain:

1. a Jacobi-Trudi type character formula in type $ABCD$, which expresses the character as a determinant of a matrix (new in CD).
2. a polyhedral multiplicity formula in type G_2, which expresses $[W : V_q(\mu)]$ as # of the lattice pts in a polyhedron.

class. lim. of minimal affinizations

Theorem (N, Li-N, '13-'15)

Assume $\mathfrak{g}$ is of type $ABCD$ or G_2, and let v be an ℓ-h.w.v. of a minimal affinization W with weight $\lambda \in \mathbb{Z}^n_{\geq 0}$.

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Def. $\mathfrak{g}[t] = \mathfrak{g} \otimes \mathbb{C}[t] \subseteq \mathcal{L}\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$: current algebra.

sketch of the poof of $U(\mathcal{L}\mathfrak{g})/U(\mathcal{L}\mathfrak{g})\, S \overset{\sim}{\to} \overline{W}$

- $U(\mathcal{L}\mathfrak{g})/U(\mathcal{L}\mathfrak{g})\, S \twoheadrightarrow \overline{W}$ is easy.
- It is enough to show ${}^{\exists}$surj. $\overline{W} \twoheadrightarrow U(\mathcal{L}\mathfrak{g})/U(\mathcal{L}\mathfrak{g})\, S$

  $\Leftrightarrow$ ${}^{\exists}\mathfrak{g}[t]$-mod. surj. $\overline{W} \twoheadrightarrow U(\mathfrak{g}[t])/U(\mathfrak{g}[t])S'$, where $S' = U(\mathfrak{g}[t]) \cap S$.

We use the theory of the affine Lie algebra $\widehat{\mathfrak{g}}$ to prove this!

$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K$: affine Lie algebra

$([X \otimes f + aK, Y \otimes g + bK] = [X, Y] \otimes fg + K(X, Y)\mathrm{Res}_{t=0}\, g \cdot df/dt)$

Rem. $\mathfrak{g}[t]$ is a Lie subalgebra of $\widehat{\mathfrak{g}}$, but $\mathcal{L}\mathfrak{g}$ is not!

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Definition

A $\mathfrak{g}[t]$-submodule $U(\mathfrak{g}[t])(v_{w_1\Lambda_1} \otimes \cdots \otimes v_{w_p\Lambda_p}) \subseteq \widehat{V}(\Lambda_1) \otimes \cdots \otimes \widehat{V}(\Lambda_p)$ is called a generalized Demazure module, where $\{v_{w\Lambda} \mid w \in \widehat{W}\}$ are distinguished vectors in $\widehat{V}(\Lambda)$ called extremal weight vectors.

Theorem (Joseph, Lakshmibai-Littelmann-Magyar)

By using the representation theory of $\widehat{\mathfrak{g}}$, we can determine defining relations of (a subclass of) generalized Demazure modules.

- Finally, show that there exist two $\mathfrak{g}[t]$-mod. surjections:

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(2) classical limit of a tensor product

$V_1, \dots, V_p$: f.d. $U_q(\mathcal{L}\mathfrak{g})$-mod. $\overset{\otimes}{\Rightarrow}$ $V_1 \otimes \cdots \otimes V_p$: f.d. $U_q(\mathcal{L}\mathfrak{g})$-mod.

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In many cases, it happens that $\overline{V_1 \otimes \cdots \otimes V_p} \not\cong \overline{V_1} \otimes \cdots \otimes \overline{V_p}$!

i.e. operations of taking tensor prod. and class. lim. are non-commutative!

Q. Can we construct $\overline{V_1 \otimes \cdots \otimes V_p}$ from $\overline{V_1}, \dots, \overline{V_p}$?

We solved this question affirmatively when $V_1, \dots, V_p$ are KR-modules.

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Theorem (N)

Assume the tensor product of KR modules $W^{i_1,\ell_1}(a_1) \otimes \cdots \otimes W^{i_p,\ell_p}(a_p)$ has a classical limit ($^\exists$sufficient conditions).

(i) If $a_1(1) = \cdots = a_p(1) =: c \in \mathbb{C}^\times$, then

$$\overline{W^{i_1,\ell_1}(a_1) \otimes \cdots \otimes W^{i_p,\ell_p}(a_p)} \cong \varphi_c^*(\overline{W^{i_1,\ell_1}(a_1)} * \cdots * \overline{W^{i_p,\ell_p}(a_p)}),$$

where $*$ is the fusion product explained below.

(ii) In the general case, we have

$$\overline{W^{i_1,\ell_1}(a_1) \otimes \cdots \otimes W^{i_p,\ell_p}(a_p)} \cong \bigotimes_{c\in\mathbb{C}^\times} \varphi_c^* \Big(\mathop{\Large *}_{k;\, a_k(1)=c} \overline{W^{i_k,\ell_k}(a_k)} \Big).$$

<u>fusion prod.</u> For $\mathfrak{g}[t]$-modules $V_1, \ldots, V_p$ (with some conditions), their fusion prod. $V_1 * \cdots * V_p$ is defined as a graded analog of tensor product, and it was defined by Feigin-Loktev as a "graded analog of the conformal coinvariants" in conformal field theory.