

量子ループ代数の有限次元表現と古典極限

直井克之 (Katsuyuki Naoi)

東京農工大学 (Tokyo University of Agriculture and Technology)

農工大数学セミナー 2017

2017年3月23日

Plan

- ① Definition of a quantum loop algebra $U_q(\mathcal{L}\mathfrak{g})$ (QLA)
- motivation, basic properties
- ② fin. dim. $U_q(\mathfrak{g})$ -modules
- ③ fin. dim. $U_q(\mathcal{L}\mathfrak{g})$ -modules
- ④ Results
 - (1) classical limits of simple modules
 - (2) classical limits of tensor products

Plan

- 1 Definition of a quantum loop algebra $U_q(\mathcal{L}\mathfrak{g})$ (QLA)
- motivation, basic properties
- 2 fin. dim. $U_q(\mathfrak{g})$ -modules
- 3 fin. dim. $U_q(\mathcal{L}\mathfrak{g})$ -modules
- 4 Results
 - (1) classical limits of simple modules
 - (2) classical limits of tensor products

Plan

- ① Definition of a quantum loop algebra $U_q(\mathcal{L}\mathfrak{g})$ (QLA)
- motivation, basic properties
- ② fin. dim. $U_q(\mathfrak{g})$ -modules
- ③ fin. dim. $U_q(\mathcal{L}\mathfrak{g})$ -modules
- ④ Results
 - (1) classical limits of simple modules
 - (2) classical limits of tensor products

Plan

- 1 Definition of a quantum loop algebra $U_q(\mathcal{L}\mathfrak{g})$ (QLA)
- motivation, basic properties
- 2 fin. dim. $U_q(\mathfrak{g})$ -modules
- 3 fin. dim. $U_q(\mathcal{L}\mathfrak{g})$ -modules
- 4 Results
 - (1) classical limits of simple modules
 - (2) classical limits of tensor products

Definition of a quantum loop algebra $U_q(\mathcal{L}\mathfrak{g})$

\mathfrak{g} : simple Lie algebra/ \mathbb{C}

(e.g. $\mathfrak{sl}_n := \{X \in \text{Mat}(n, \mathbb{C}) \mid \text{tr } X = 0\}$, $[X, Y] = XY - YX$)

$\mathcal{L}\mathfrak{g} := \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$: loop algebra ($[X \otimes f, Y \otimes g] = [X, Y] \otimes fg$)

Quantum loop algebra $U_q(\mathcal{L}\mathfrak{g})$ (QLA)

= the **quantized enveloping algebra** assoc. with $\mathcal{L}\mathfrak{g}$.

\mathfrak{a} (Lie algebra such as \mathfrak{g} , $\mathcal{L}\mathfrak{g}$)

$\implies U(\mathfrak{a})$: universal enveloping algebra of \mathfrak{a} (associative \mathbb{C} -algebra)

$\xrightarrow{q\text{-deform}} U_q(\mathfrak{a})$: the quantized env. algebra (associative $\mathbb{C}(q)$ -algebra)

Definition of a quantum loop algebra $U_q(\mathcal{L}\mathfrak{g})$

\mathfrak{g} : simple Lie algebra/ \mathbb{C}

(e.g. $\mathfrak{sl}_n := \{X \in \text{Mat}(n, \mathbb{C}) \mid \text{tr } X = 0\}$, $[X, Y] = XY - YX$)

$\mathcal{L}\mathfrak{g} := \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$: loop algebra ($[X \otimes f, Y \otimes g] = [X, Y] \otimes fg$)

Quantum loop algebra $U_q(\mathcal{L}\mathfrak{g})$ (QLA)

= the **quantized enveloping algebra** assoc. with $\mathcal{L}\mathfrak{g}$.

\mathfrak{a} (Lie algebra such as \mathfrak{g} , $\mathcal{L}\mathfrak{g}$)

$\implies U(\mathfrak{a})$: universal enveloping algebra of \mathfrak{a} (associative \mathbb{C} -algebra)

$\xrightarrow{q\text{-deform}} U_q(\mathfrak{a})$: the quantized env. algebra (associative $\mathbb{C}(q)$ -algebra)

Definition of a quantum loop algebra $U_q(\mathcal{L}\mathfrak{g})$

\mathfrak{g} : simple Lie algebra/ \mathbb{C}

(e.g. $\mathfrak{sl}_n := \{X \in \text{Mat}(n, \mathbb{C}) \mid \text{tr } X = 0\}$, $[X, Y] = XY - YX$)

$\mathcal{L}\mathfrak{g} := \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$: loop algebra ($[X \otimes f, Y \otimes g] = [X, Y] \otimes fg$)

Quantum loop algebra $U_q(\mathcal{L}\mathfrak{g})$ (QLA)

= the **quantized enveloping algebra** assoc. with $\mathcal{L}\mathfrak{g}$.

\mathfrak{a} (Lie algebra such as \mathfrak{g} , $\mathcal{L}\mathfrak{g}$)

$\implies U(\mathfrak{a})$: universal enveloping algebra of \mathfrak{a} (associative \mathbb{C} -algebra)

$\xrightarrow{q\text{-deform}} U_q(\mathfrak{a})$: the quantized env. algebra (associative $\mathbb{C}(q)$ -algebra)

universal enveloping algebra

First step \mathfrak{a} : Lie algebra $\Rightarrow U(\mathfrak{a})$: universal enveloping algebra

$$U(\mathfrak{a}) := T(\mathfrak{a}) / \langle XY - YX - [X, Y] \mid X, Y \in \mathfrak{a} \rangle$$

$$(T(\mathfrak{a}) := \bigoplus_{k=0}^{\infty} T^k(\mathfrak{a})): \text{ tensor algebra of } \mathfrak{a},$$

i.e. the free (associative) algebra generated by \mathfrak{a})

Rem. V is an \mathfrak{a} -module $\Leftrightarrow V$ is a $U(\mathfrak{a})$ -module

(c.f. group algebra $\mathbb{C}[G]$ for a group G)

universal enveloping algebra

First step \mathfrak{a} : Lie algebra $\Rightarrow U(\mathfrak{a})$: universal enveloping algebra

$$U(\mathfrak{a}) := T(\mathfrak{a}) / \langle XY - YX - [X, Y] \mid X, Y \in \mathfrak{a} \rangle$$

$$(T(\mathfrak{a}) := \bigoplus_{k=0}^{\infty} T^k(\mathfrak{a})): \text{ tensor algebra of } \mathfrak{a},$$

i.e. the free (associative) algebra generated by \mathfrak{a})

Rem. V is an \mathfrak{a} -module $\Leftrightarrow V$ is a $U(\mathfrak{a})$ -module

(c.f. group algebra $\mathbb{C}[G]$ for a group G)

quantized enveloping algebra

Second step $U(\mathfrak{a})$: univ. env. alg. $\xrightarrow{q\text{-defom}}$ $U_q(\mathfrak{a})$: quantized env. alg.

$U_q(\mathfrak{a})$ is defined as a q -deformation of $U(\mathfrak{a})$ i.e. " $\lim_{q \rightarrow 1} U_q(\mathfrak{a}) = U(\mathfrak{a})$ ".

Ex.1 $\mathfrak{a} = \mathfrak{sl}_2 = \mathbb{C} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \oplus \mathbb{C} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \oplus \mathbb{C} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \mathbb{C}e \oplus \mathbb{C}h \oplus \mathbb{C}f$

$$[e, f] = h, \quad [h, e] = 2e, \quad [h, f] = -2f$$

$U(\mathfrak{sl}_2)$: the \mathbb{C} -algebra generated by e, h, f with relations:

$$ef - fe = h, \quad he - eh = 2e, \quad hf - fh = -2f \quad (*)$$

(Recall $U(\mathfrak{a}) := T(\mathfrak{a}) / \langle XY - YX - [X, Y] \mid X, Y \in \mathfrak{a} \rangle$)

$U_q(\mathfrak{sl}_2)$: the $\mathbb{C}(q)$ -algebra gene'd by $e, q^{\pm h}, f$ with relations:

$$q^h q^{-h} = q^{-h} q^h = 1, \quad ef - fe = \frac{q^h - q^{-h}}{q - q^{-1}}, \quad q^h e q^{-h} = q^2 e, \quad q^h f q^{-h} = q^{-2} f$$

($q = 1 + t, q^h = 1 + th, t \rightarrow 0$ recovers $(*)$)

quantized enveloping algebra

Second step $U(\mathfrak{a})$: univ. env. alg. $\xrightarrow{q\text{-defom}}$ $U_q(\mathfrak{a})$: quantized env. alg.

$U_q(\mathfrak{a})$ is defined as a q -deformation of $U(\mathfrak{a})$ i.e. “ $\lim_{q \rightarrow 1} U_q(\mathfrak{a}) = U(\mathfrak{a})$ ”.

Ex.1 $\mathfrak{a} = \mathfrak{sl}_2 = \mathbb{C} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \oplus \mathbb{C} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \oplus \mathbb{C} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \mathbb{C}e \oplus \mathbb{C}h \oplus \mathbb{C}f$

$$[e, f] = h, \quad [h, e] = 2e, \quad [h, f] = -2f$$

$U(\mathfrak{sl}_2)$: the \mathbb{C} -algebra generated by e, h, f with relations:

$$ef - fe = h, \quad he - eh = 2e, \quad hf - fh = -2f \quad - (*)$$

(Recall $U(\mathfrak{a}) := T(\mathfrak{a}) / \langle XY - YX - [X, Y] \mid X, Y \in \mathfrak{a} \rangle$)

$U_q(\mathfrak{sl}_2)$: the $\mathbb{C}(q)$ -algebra gene'd by $e, q^{\pm h}, f$ with relations:

$$q^h q^{-h} = q^{-h} q^h = 1, \quad ef - fe = \frac{q^h - q^{-h}}{q - q^{-1}}, \quad q^h e q^{-h} = q^2 e, \quad q^h f q^{-h} = q^{-2} f$$

($q = 1 + t, q^h = 1 + th, t \rightarrow 0$ recovers $(*)$)

quantized enveloping algebra

Second step $U(\mathfrak{a})$: univ. env. alg. $\xrightarrow{q\text{-defom}}$ $U_q(\mathfrak{a})$: quantized env. alg.

$U_q(\mathfrak{a})$ is defined as a q -deformation of $U(\mathfrak{a})$ i.e. " $\lim_{q \rightarrow 1} U_q(\mathfrak{a}) = U(\mathfrak{a})$ ".

Ex.1 $\mathfrak{a} = \mathfrak{sl}_2 = \mathbb{C} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \oplus \mathbb{C} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \oplus \mathbb{C} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \mathbb{C}e \oplus \mathbb{C}h \oplus \mathbb{C}f$

$$[e, f] = h, \quad [h, e] = 2e, \quad [h, f] = -2f$$

$U(\mathfrak{sl}_2)$: the \mathbb{C} -algebra generated by e, h, f with relations:

$$ef - fe = h, \quad he - eh = 2e, \quad hf - fh = -2f \quad - (*)$$

(Recall $U(\mathfrak{a}) := T(\mathfrak{a}) / \langle XY - YX - [X, Y] \mid X, Y \in \mathfrak{a} \rangle$)

$U_q(\mathfrak{sl}_2)$: the $\mathbb{C}(q)$ -algebra gene'd by $e, q^{\pm h}, f$ with relations:

$$q^h q^{-h} = q^{-h} q^h = 1, \quad ef - fe = \frac{q^h - q^{-h}}{q - q^{-1}}, \quad q^h e q^{-h} = q^2 e, \quad q^h f q^{-h} = q^{-2} f$$

($q = 1 + t, q^h = 1 + th, t \rightarrow 0$ recovers $(*)$)

quantized enveloping algebra

Second step $U(\mathfrak{a})$: univ. env. alg. $\xrightarrow{q\text{-defom}}$ $U_q(\mathfrak{a})$: quantized env. alg.

$U_q(\mathfrak{a})$ is defined as a q -deformation of $U(\mathfrak{a})$ i.e. “ $\lim_{q \rightarrow 1} U_q(\mathfrak{a}) = U(\mathfrak{a})$ ”.

Ex.1 $\mathfrak{a} = \mathfrak{sl}_2 = \mathbb{C} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \oplus \mathbb{C} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \oplus \mathbb{C} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \mathbb{C}e \oplus \mathbb{C}h \oplus \mathbb{C}f$

$$[e, f] = h, \quad [h, e] = 2e, \quad [h, f] = -2f$$

$U(\mathfrak{sl}_2)$: the \mathbb{C} -algebra generated by e, h, f with relations:

$$ef - fe = h, \quad he - eh = 2e, \quad hf - fh = -2f \quad - (*)$$

(Recall $U(\mathfrak{a}) := T(\mathfrak{a}) / \langle XY - YX - [X, Y] \mid X, Y \in \mathfrak{a} \rangle$)

$U_q(\mathfrak{sl}_2)$: the $\mathbb{C}(q)$ -algebra gene'd by $e, q^{\pm h}, f$ with relations:

$$q^h q^{-h} = q^{-h} q^h = 1, \quad ef - fe = \frac{q^h - q^{-h}}{q - q^{-1}}, \quad q^h e q^{-h} = q^2 e, \quad q^h f q^{-h} = q^{-2} f$$

($q = 1 + t, q^h = 1 + th, t \rightarrow 0$ recovers $(*)$)

Ex.2 $\mathfrak{a} = \mathcal{L}\mathfrak{sl}_2 (= \mathfrak{sl}_2 \otimes \mathbb{C}[t^{\pm 1}])$

$U(\mathcal{L}\mathfrak{sl}_2)$: the \mathbb{C} -algebra generated by $e, h, f, \tilde{e}, \tilde{f}$ with rel.:

- e, h, f satisfy the rel. in $U(\mathfrak{sl}_2)$, ○ \tilde{e}, h, \tilde{f} satisfy the rel. in $U(\mathfrak{sl}_2)$,
- $\tilde{e}^3 f - 3\tilde{e}^2 f \tilde{e} + 3\tilde{e} f \tilde{e}^2 - f \tilde{e}^3 = 0$, $f^3 \tilde{e} - 3f^2 \tilde{e} f + 3f \tilde{e} f^2 - \tilde{e} f^3 = 0$,
- The same with \tilde{e}, f replaced by \tilde{f}, e

$U_q(\mathcal{L}\mathfrak{sl}_2)$: the $\mathbb{C}(q)$ -alg. gene'd by $e, q^{\pm h}, f, \tilde{e}, \tilde{f}$ with rel.:

- $e, q^{\pm h}, f$ satisfy the rel. in $U_q(\mathfrak{sl}_2)$, ○ $\tilde{e}, q^{\pm h}, \tilde{f}$ satisfy the rel. in $U_q(\mathfrak{sl}_2)$,
- $\tilde{e}^3 f - [3]\tilde{e}^2 f \tilde{e} + [3]\tilde{e} f \tilde{e}^2 - f \tilde{e}^3 = 0$, $f^3 \tilde{e} - [3]f^2 \tilde{e} f + [3]f \tilde{e} f^2 - \tilde{e} f^3 = 0$
($[3] = q^2 + 1 + q^{-2}$),
- The same with \tilde{e}, f replaced by \tilde{f}, e

$\mathcal{L}\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}[t^{\pm 1}]$ (\mathfrak{g} : simple Lie alg.) $\rightsquigarrow U_q(\mathcal{L}\mathfrak{g})$: quantum loop alg.

Ex.2 $\mathfrak{a} = \mathcal{L}\mathfrak{sl}_2 (= \mathfrak{sl}_2 \otimes \mathbb{C}[t^{\pm 1}])$

$U(\mathcal{L}\mathfrak{sl}_2)$: the \mathbb{C} -algebra generated by $e, h, f, \tilde{e}, \tilde{f}$ with rel.:

- e, h, f satisfy the rel. in $U(\mathfrak{sl}_2)$, ○ \tilde{e}, h, \tilde{f} satisfy the rel. in $U(\mathfrak{sl}_2)$,
- $\tilde{e}^3 f - 3\tilde{e}^2 f \tilde{e} + 3\tilde{e} f \tilde{e}^2 - f \tilde{e}^3 = 0$, $f^3 \tilde{e} - 3f^2 \tilde{e} f + 3f \tilde{e} f^2 - \tilde{e} f^3 = 0$,
- The same with \tilde{e}, f replaced by \tilde{f}, e

$U_q(\mathcal{L}\mathfrak{sl}_2)$: the $\mathbb{C}(q)$ -alg. gene'd by $e, q^{\pm h}, f, \tilde{e}, \tilde{f}$ with rel.:

- $e, q^{\pm h}, f$ satisfy the rel. in $U_q(\mathfrak{sl}_2)$, ○ $\tilde{e}, q^{\pm h}, \tilde{f}$ satisfy the rel. in $U_q(\mathfrak{sl}_2)$,
- $\tilde{e}^3 f - [3]\tilde{e}^2 f \tilde{e} + [3]\tilde{e} f \tilde{e}^2 - f \tilde{e}^3 = 0$, $f^3 \tilde{e} - [3]f^2 \tilde{e} f + [3]f \tilde{e} f^2 - \tilde{e} f^3 = 0$
 ($[3] = q^2 + 1 + q^{-2}$),
- The same with \tilde{e}, f replaced by \tilde{f}, e

$\mathcal{L}\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}[t^{\pm 1}]$ (\mathfrak{g} : simple Lie alg.) $\rightsquigarrow U_q(\mathcal{L}\mathfrak{g})$: quantum loop alg.

Ex.2 $\mathfrak{a} = \mathcal{L}\mathfrak{sl}_2 (= \mathfrak{sl}_2 \otimes \mathbb{C}[t^{\pm 1}])$

$U(\mathcal{L}\mathfrak{sl}_2)$: the \mathbb{C} -algebra generated by $e, h, f, \tilde{e}, \tilde{f}$ with rel.:

- e, h, f satisfy the rel. in $U(\mathfrak{sl}_2)$, ○ \tilde{e}, h, \tilde{f} satisfy the rel. in $U(\mathfrak{sl}_2)$,
- $\tilde{e}^3 f - 3\tilde{e}^2 f \tilde{e} + 3\tilde{e} f \tilde{e}^2 - f \tilde{e}^3 = 0, \quad f^3 \tilde{e} - 3f^2 \tilde{e} f + 3f \tilde{e} f^2 - \tilde{e} f^3 = 0,$
- The same with \tilde{e}, f replaced by \tilde{f}, e

$U_q(\mathcal{L}\mathfrak{sl}_2)$: the $\mathbb{C}(q)$ -alg. gene'd by $e, q^{\pm h}, f, \tilde{e}, \tilde{f}$ with rel.:

- $e, q^{\pm h}, f$ satisfy the rel. in $U_q(\mathfrak{sl}_2)$, ○ $\tilde{e}, q^{\pm h}, \tilde{f}$ satisfy the rel. in $U_q(\mathfrak{sl}_2)$,
- $\tilde{e}^3 f - [3]\tilde{e}^2 f \tilde{e} + [3]\tilde{e} f \tilde{e}^2 - f \tilde{e}^3 = 0, \quad f^3 \tilde{e} - [3]f^2 \tilde{e} f + [3]f \tilde{e} f^2 - \tilde{e} f^3 = 0$
 ($[3] = q^2 + 1 + q^{-2}$),
- The same with \tilde{e}, f replaced by \tilde{f}, e

$\mathcal{L}\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}[t^{\pm 1}]$ (\mathfrak{g} : simple Lie alg.) $\rightsquigarrow U_q(\mathcal{L}\mathfrak{g})$: quantum loop alg.

Ex.2 $\mathfrak{a} = \mathcal{L}\mathfrak{sl}_2 (= \mathfrak{sl}_2 \otimes \mathbb{C}[t^{\pm 1}])$

$U(\mathcal{L}\mathfrak{sl}_2)$: the \mathbb{C} -algebra generated by $e, h, f, \tilde{e}, \tilde{f}$ with rel.:

- e, h, f satisfy the rel. in $U(\mathfrak{sl}_2)$,
- \tilde{e}, h, \tilde{f} satisfy the rel. in $U(\mathfrak{sl}_2)$,
- $\tilde{e}^3 f - 3\tilde{e}^2 f \tilde{e} + 3\tilde{e} f \tilde{e}^2 - f \tilde{e}^3 = 0, f^3 \tilde{e} - 3f^2 \tilde{e} f + 3f \tilde{e} f^2 - \tilde{e} f^3 = 0,$
- The same with \tilde{e}, f replaced by \tilde{f}, e

$U_q(\mathcal{L}\mathfrak{sl}_2)$: the $\mathbb{C}(q)$ -alg. gene'd by $e, q^{\pm h}, f, \tilde{e}, \tilde{f}$ with rel.:

- $e, q^{\pm h}, f$ satisfy the rel. in $U_q(\mathfrak{sl}_2)$,
- $\tilde{e}, q^{\pm h}, \tilde{f}$ satisfy the rel. in $U_q(\mathfrak{sl}_2)$,
- $\tilde{e}^3 f - [3]\tilde{e}^2 f \tilde{e} + [3]\tilde{e} f \tilde{e}^2 - f \tilde{e}^3 = 0, f^3 \tilde{e} - [3]f^2 \tilde{e} f + [3]f \tilde{e} f^2 - \tilde{e} f^3 = 0$
($[3] = q^2 + 1 + q^{-2}$),
- The same with \tilde{e}, f replaced by \tilde{f}, e

$\mathcal{L}\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}[t^{\pm 1}]$ (\mathfrak{g} : simple Lie alg.) $\rightsquigarrow U_q(\mathcal{L}\mathfrak{g})$: **quantum loop alg.**

Q. There are any number of algebras which specialize to $U(\mathfrak{a})$ at $q = 1$. Why the q -derom. $U_q(\mathfrak{a})$ is particularly important?

Note $U(\mathfrak{a})$ is a **Hopf algebra!**

i.e. \exists coproduct $\Delta: U(\mathfrak{a}) \rightarrow U(\mathfrak{a}) \otimes U(\mathfrak{a})$, \exists counit $\varepsilon: U(\mathfrak{a}) \rightarrow \mathbb{C}$,

\exists antipode $S: U(\mathfrak{a}) \xrightarrow{\sim} U(\mathfrak{a})^{\text{op}}$ with some compatibility.

(\Rightarrow If V, W are $U(\mathfrak{a})$ -mod., then $V \otimes W, \mathbb{C}, V^*$ become $U(\mathfrak{a})$ -modules.)

Theorem

$U_q(\mathfrak{a})$ has a Hopf algebra structure.

That is, $U_q(\mathfrak{a})$ is q -deform. of $U(\mathfrak{a})$ as a Hopf algebra!

Rem. Moreover, $U_q(\mathfrak{a})$ is a **quasi-triangular Hopf alg.**, which implies that a solution (**R-matrix**) of the Yang-Baxter equation can be obtained from a pair of $U_q(\mathfrak{a})$ -modules.

Q. There are any number of algebras which specialize to $U(\mathfrak{a})$ at $q = 1$. Why the q -derom. $U_q(\mathfrak{a})$ is particularly important?

Note $U(\mathfrak{a})$ is a **Hopf algebra!**

i.e. \exists coproduct $\Delta: U(\mathfrak{a}) \rightarrow U(\mathfrak{a}) \otimes U(\mathfrak{a})$, \exists counit $\varepsilon: U(\mathfrak{a}) \rightarrow \mathbb{C}$,

\exists antipode $S: U(\mathfrak{a}) \xrightarrow{\sim} U(\mathfrak{a})^{\text{op}}$ with some compatibility.

(\Rightarrow If V, W are $U(\mathfrak{a})$ -mod., then $V \otimes W, \mathbb{C}, V^*$ become $U(\mathfrak{a})$ -modules.)

Theorem

$U_q(\mathfrak{a})$ has a Hopf algebra structure.

That is, $U_q(\mathfrak{a})$ is q -deform. of $U(\mathfrak{a})$ as a Hopf algebra!

Rem. Moreover, $U_q(\mathfrak{a})$ is a **quasi-triangular Hopf alg.**, which implies that a solution (**R-matrix**) of the Yang-Baxter equation can be obtained from a pair of $U_q(\mathfrak{a})$ -modules.

Q. There are any number of algebras which specialize to $U(\mathfrak{a})$ at $q = 1$. Why the q -derom. $U_q(\mathfrak{a})$ is particularly important?

Note $U(\mathfrak{a})$ is a **Hopf algebra!**

i.e. \exists coproduct $\Delta: U(\mathfrak{a}) \rightarrow U(\mathfrak{a}) \otimes U(\mathfrak{a})$, \exists counit $\varepsilon: U(\mathfrak{a}) \rightarrow \mathbb{C}$,

\exists antipode $S: U(\mathfrak{a}) \xrightarrow{\sim} U(\mathfrak{a})^{\text{op}}$ with some compatibility.

(\Rightarrow If V, W are $U(\mathfrak{a})$ -mod., then $V \otimes W, \mathbb{C}, V^*$ become $U(\mathfrak{a})$ -modules.)

Theorem

$U_q(\mathfrak{a})$ has a Hopf algebra structure.

That is, $U_q(\mathfrak{a})$ is q -deform. of $U(\mathfrak{a})$ as a Hopf algebra!

Rem. Moreover, $U_q(\mathfrak{a})$ is a **quasi-triangular Hopf alg.**, which implies that a solution (**R-matrix**) of the Yang-Baxter equation can be obtained from a pair of $U_q(\mathfrak{a})$ -modules.

Q. There are any number of algebras which specialize to $U(\mathfrak{a})$ at $q = 1$. Why the q -derom. $U_q(\mathfrak{a})$ is particularly important?

Note $U(\mathfrak{a})$ is a **Hopf algebra!**

i.e. \exists coproduct $\Delta: U(\mathfrak{a}) \rightarrow U(\mathfrak{a}) \otimes U(\mathfrak{a})$, \exists counit $\varepsilon: U(\mathfrak{a}) \rightarrow \mathbb{C}$,

\exists antipode $S: U(\mathfrak{a}) \xrightarrow{\sim} U(\mathfrak{a})^{\text{op}}$ with some compatibility.

(\Rightarrow If V, W are $U(\mathfrak{a})$ -mod., then $V \otimes W, \mathbb{C}, V^*$ become $U(\mathfrak{a})$ -modules.)

Theorem

$U_q(\mathfrak{a})$ has a Hopf algebra structure.

That is, $U_q(\mathfrak{a})$ is q -deform. of $U(\mathfrak{a})$ as a Hopf algebra!

Rem. Moreover, $U_q(\mathfrak{a})$ is a **quasi-triangular Hopf alg.**, which implies that a solution (**R-matrix**) of the Yang-Baxter equation can be obtained from a pair of $U_q(\mathfrak{a})$ -modules.

Motivation to the study of fin. dim. $U_q(\mathcal{L}\mathfrak{g})$ -mod.

- 1 All the $U_q(\mathcal{L}\mathfrak{g})$ -modules are too large to controll.
- 2 f.d. $U_q(\mathcal{L}\mathfrak{g})$ -mod. have rich structures:
 - The category of f.d. $U_q(\mathcal{L}\mathfrak{g})$ -mod. is a monoidal category via \otimes
(Recall $U_q(\mathcal{L}\mathfrak{g})$: Hopf alg. $\Rightarrow V \otimes W$: $U_q(\mathcal{L}\mathfrak{g})$ -mod.)
 - not semisimple ○ uncountable simple modules
- 3 have connection to other fields:
 - integrable system
 - geometry on quiver variety (instanton moduli on ALE space)
 - combinatorics, cluster algebra, . . . , etc.
- 4 many problems remain unsolved (str. of simple mod., tensor prod.)
c.f., f.d. $U_q(\mathfrak{g})$ -mod. are deeply understood ← next slides

Motivation to the study of fin. dim. $U_q(\mathcal{L}\mathfrak{g})$ -mod.

- 1 All the $U_q(\mathcal{L}\mathfrak{g})$ -modules are too large to controll.
- 2 f.d. $U_q(\mathcal{L}\mathfrak{g})$ -mod. have rich structures:
 - The category of f.d. $U_q(\mathcal{L}\mathfrak{g})$ -mod. is a monoidal category via \otimes
(Recall $U_q(\mathcal{L}\mathfrak{g})$: Hopf alg. $\Rightarrow V \otimes W$: $U_q(\mathcal{L}\mathfrak{g})$ -mod.)
 - not semisimple ○ uncountable simple modules
- 3 have connection to other fields:
 - integrable system
 - geometry on quiver variety (instanton moduli on ALE space)
 - combinatorics, cluster algebra, \dots , etc.
- 4 many problems remain unsolved (str. of simple mod., tensor prod.)
c.f., f.d. $U_q(\mathfrak{g})$ -mod. are deeply understood \leftarrow next slides

Motivation to the study of fin. dim. $U_q(\mathcal{L}\mathfrak{g})$ -mod.

- 1 All the $U_q(\mathcal{L}\mathfrak{g})$ -modules are too large to controll.
- 2 f.d. $U_q(\mathcal{L}\mathfrak{g})$ -mod. have rich structures:
 - The category of f.d. $U_q(\mathcal{L}\mathfrak{g})$ -mod. is a monoidal category via \otimes
(Recall $U_q(\mathcal{L}\mathfrak{g})$: Hopf alg. $\Rightarrow V \otimes W$: $U_q(\mathcal{L}\mathfrak{g})$ -mod.)
 - not semisimple ○ uncountable simple modules
- 3 have connection to other fields:
 - integrable system
 - geometry on quiver variety (instanton moduli on ALE space)
 - combinatorics, cluster algebra, . . . , etc.
- 4 many problems remain unsolved (str. of simple mod., tensor prod.)
c.f., f.d. $U_q(\mathfrak{g})$ -mod. are deeply understood ← next slides

Motivation to the study of fin. dim. $U_q(\mathcal{L}\mathfrak{g})$ -mod.

- 1 All the $U_q(\mathcal{L}\mathfrak{g})$ -modules are too large to controll.
- 2 f.d. $U_q(\mathcal{L}\mathfrak{g})$ -mod. have rich structures:
 - The category of f.d. $U_q(\mathcal{L}\mathfrak{g})$ -mod. is a monoidal category via \otimes
(Recall $U_q(\mathcal{L}\mathfrak{g})$: Hopf alg. $\Rightarrow V \otimes W$: $U_q(\mathcal{L}\mathfrak{g})$ -mod.)
 - not semisimple ○ uncountable simple modules
- 3 have connection to other fields:
 - integrable system
 - geometry on quiver variety (instanton moduli on ALE space)
 - combinatorics, cluster algebra, . . . , etc.
- 4 many problems remain unsolved (str. of simple mod., tensor prod.)
c.f., f.d. $U_q(\mathfrak{g})$ -mod. are deeply understood ← next slides

fin. dim. $U_q(\mathfrak{g})$ -modules

Proposition

① f.d. $U_q(\mathfrak{g})$ -mod. are semisimple (i.e. $\forall V \cong \bigoplus(\text{simple})$)

② {f.d. simple $U_q(\mathfrak{g})$ -mod.} $\stackrel{1-1}{\leftrightarrow} \mathbb{Z}_{\geq 0}^n$

$$V_q(\lambda) \leftrightarrow \lambda = (\lambda_1, \dots, \lambda_n)$$

Q. ○ Study the structure of simple mod. $V_q(\lambda)$, such as character $\text{ch } V_q(\lambda)$.

($V = \bigoplus_{\alpha} V_{\alpha}$: simultaneous eigensp. dec. w.r.t. a comm. subalg.)

$$\Rightarrow \text{ch } V := \sum_{\alpha} t^{\alpha} \dim V_{\alpha} \in \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$$

○ Given two simple $U_q(\mathfrak{g})$ -mod. $V_q(\lambda)$, $V_q(\mu)$, determine the multiplicities $[V_q(\lambda) \otimes V_q(\mu) : V_q(\nu)]$ for every ν .

fin. dim. $U_q(\mathfrak{g})$ -modules

Proposition

① f.d. $U_q(\mathfrak{g})$ -mod. are semisimple (i.e. $\forall V \cong \bigoplus(\text{simple})$)

② {f.d. simple $U_q(\mathfrak{g})$ -mod.} $\xleftrightarrow{1-1} \mathbb{Z}_{\geq 0}^n$

$$V_q(\lambda) \leftrightarrow \lambda = (\lambda_1, \dots, \lambda_n)$$

Q. ○ Study the structure of simple mod. $V_q(\lambda)$, such as character $\text{ch } V_q(\lambda)$.

($V = \bigoplus_{\alpha} V_{\alpha}$: simultaneous eigensp. dec. w.r.t. a comm. subalg.)

$$\Rightarrow \text{ch } V := \sum_{\alpha} t^{\alpha} \dim V_{\alpha} \in \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$$

○ Given two simple $U_q(\mathfrak{g})$ -mod. $V_q(\lambda)$, $V_q(\mu)$, determine the multiplicities $[V_q(\lambda) \otimes V_q(\mu) : V_q(\nu)]$ for every ν .

Definition (Classical limit)

V : f.d. $U_q(\mathfrak{g})$ -mod. $\xrightarrow{\lim_{q \rightarrow 1}} \bar{V}$: f.d. $U(\mathfrak{g})$ -mod. (= \mathfrak{g} -mod.)

(Recall “ $\lim_{q \rightarrow 1} U_q(\mathfrak{g}) = U(\mathfrak{g})$ ”)

Proposition

- 1 For every $\lambda \in \mathbb{Z}_{\geq 0}^n$, $V(\lambda) := \overline{V_q(\lambda)}$ is a simple \mathfrak{g} -mod., and $V_q(\lambda) \mapsto V(\lambda)$ induces a bij. from {f.d. simple $U_q(\mathfrak{g})$ -mod.} to {f.d. simple \mathfrak{g} -mod.}

($\Rightarrow \text{ch } V_q(\lambda) = \text{ch } V(\lambda) \leftarrow$ Weyl's character formula)

- 2 For all $\lambda, \mu, \nu \in \mathbb{Z}_{\geq 0}^n$,

$$[V_q(\lambda) \otimes V_q(\mu) : V_q(\nu)] = [V(\lambda) \otimes V(\mu) : V(\nu)]$$



combinatorial formulas

Definition (Classical limit)

V : f.d. $U_q(\mathfrak{g})$ -mod. $\xrightarrow{\lim_{q \rightarrow 1}} \bar{V}$: f.d. $U(\mathfrak{g})$ -mod. (= \mathfrak{g} -mod.)

(Recall “ $\lim_{q \rightarrow 1} U_q(\mathfrak{g}) = U(\mathfrak{g})$ ”)

Proposition

- ① For every $\lambda \in \mathbb{Z}_{\geq 0}^n$, $V(\lambda) := \overline{V_q(\lambda)}$ is a simple \mathfrak{g} -mod., and $V_q(\lambda) \mapsto V(\lambda)$ induces a bij. from {f.d. simple $U_q(\mathfrak{g})$ -mod.} to {f.d. simple \mathfrak{g} -mod.}

($\Rightarrow \text{ch } V_q(\lambda) = \text{ch } V(\lambda)$ ← **Weyl's character formula**)

- ② For all $\lambda, \mu, \nu \in \mathbb{Z}_{\geq 0}^n$,

$$[V_q(\lambda) \otimes V_q(\mu) : V_q(\nu)] = [V(\lambda) \otimes V(\mu) : V(\nu)]$$

↑
combinatorial formulas

fin. dim. $U_q(\mathcal{L}\mathfrak{g})$ -modules

Theorem (Chari-Pressley, 95)

 $\{\text{f.d. simple } U_q(\mathcal{L}\mathfrak{g})\text{-mod.}\}$

$$\xleftrightarrow{1-1} \{\boldsymbol{\pi} = (\pi_1(u), \dots, \pi_n(u)) \mid \pi_k(u) \in 1 + u\mathbb{C}(q)[u]\}$$

Denote the simples by $L_q(\boldsymbol{\pi})$ ($\boldsymbol{\pi}$ are called the **Drinfeld polynomials**).

Rem. f.d. $U_q(\mathcal{L}\mathfrak{g})$ -mod. are not semisimple.

Q. (i) Study the structure of $L_q(\boldsymbol{\pi})$.

(ii) Study $L_q(\boldsymbol{\pi}) \otimes L_q(\boldsymbol{\pi}')$.

Rem. By the geometric rep. theory on quiver varieties, H. Nakajima has obtained an algorithm to give characters (or q -characters) of every $L_q(\boldsymbol{\pi})$. The algorithm is very complicated, however, and a closed formula is still to be established (at least to some “good class” of simple modules).

fin. dim. $U_q(\mathcal{L}\mathfrak{g})$ -modules

Theorem (Chari-Pressley, 95)

{f.d. simple $U_q(\mathcal{L}\mathfrak{g})$ -mod.}

$$\xleftrightarrow{1-1} \{ \boldsymbol{\pi} = (\pi_1(u), \dots, \pi_n(u)) \mid \pi_k(u) \in 1 + u\mathbb{C}(q)[u] \}$$

Denote the simples by $L_q(\boldsymbol{\pi})$ ($\boldsymbol{\pi}$ are called the **Drinfeld polynomials**).

Rem. f.d. $U_q(\mathcal{L}\mathfrak{g})$ -mod. are not semisimple.

Q. (i) Study the structure of $L_q(\boldsymbol{\pi})$.

(ii) Study $L_q(\boldsymbol{\pi}) \otimes L_q(\boldsymbol{\pi}')$.

Rem. By the geometric rep. theory on quiver varieties, H. Nakajima has obtained an algorithm to give characters (or q -characters) of every $L_q(\boldsymbol{\pi})$. The algorithm is very complicated, however, and a closed formula is still to be established (at least to some “good class” of simple modules).

fin. dim. $U_q(\mathcal{L}\mathfrak{g})$ -modules

Theorem (Chari-Pressley, 95)

{f.d. simple $U_q(\mathcal{L}\mathfrak{g})$ -mod.}

$$\xleftrightarrow{1-1} \{ \boldsymbol{\pi} = (\pi_1(u), \dots, \pi_n(u)) \mid \pi_k(u) \in 1 + u\mathbb{C}(q)[u] \}$$

Denote the simples by $L_q(\boldsymbol{\pi})$ ($\boldsymbol{\pi}$ are called the **Drinfeld polynomials**).

Rem. f.d. $U_q(\mathcal{L}\mathfrak{g})$ -mod. are not semisimple.

Q. (i) Study the structure of $L_q(\boldsymbol{\pi})$.

(ii) Study $L_q(\boldsymbol{\pi}) \otimes L_q(\boldsymbol{\pi}')$.

Rem. By the geometric rep. theory on quiver varieties, H. Nakajima has obtained an algorithm to give characters (or q -characters) of every $L_q(\boldsymbol{\pi})$. The algorithm is very complicated, however, and a closed formula is still to be established (at least to some “good class” of simple modules).

classical limits of f.d. $U_q(\mathcal{L}\mathfrak{g})$ -modules

V : f.d. $U_q(\mathcal{L}\mathfrak{g})$ -mod. with some conditions (e.g. almost all simples)

$\lim_{q \rightarrow 1} \bar{V}$: $U(\mathcal{L}\mathfrak{g})$ -mod. = $\mathcal{L}\mathfrak{g}$ -mod.

Goal Study the structure of V by observing \bar{V} instead.

good point ○ $U(\mathcal{L}\mathfrak{g})$ is a much simpler alg. than $U_q(\mathcal{L}\mathfrak{g})$.

○ $\mathfrak{g} = \mathfrak{g} \otimes 1 \subseteq \mathfrak{g} \otimes \mathbb{C}[t^{\pm 1}] = \mathcal{L}\mathfrak{g}$: Lie subalg., $U_q(\mathfrak{g}) \subseteq U_q(\mathcal{L}\mathfrak{g})$: subalg.,

and $V \cong_{U_q(\mathfrak{g})} \bigoplus_{\lambda} V_q(\lambda)^{\oplus m_{\lambda}} \Leftrightarrow \bar{V} \cong_{\mathfrak{g}} \bigoplus_{\lambda} V(\lambda)^{\oplus m_{\lambda}}$,

that is, taking the classical limit does not lose any information of the $U_q(\mathfrak{g})$ -module structure (in particular, $\text{ch } V = \text{ch } \bar{V}$)

bad point

f.d. $\mathcal{L}\mathfrak{g}$ -modules are not semisimple, and even V is simple, \bar{V} is

rarely simple \Rightarrow need to treat a nonsimple, indec. $\mathcal{L}\mathfrak{g}$ -mod.

classical limits of f.d. $U_q(\mathcal{L}\mathfrak{g})$ -modules

V : f.d. $U_q(\mathcal{L}\mathfrak{g})$ -mod. with some conditions (e.g. almost all simples)

$\lim_{q \rightarrow 1} \bar{V}: U(\mathcal{L}\mathfrak{g})$ -mod. = $\mathcal{L}\mathfrak{g}$ -mod.

Goal Study the structure of V by observing \bar{V} instead.

good point ○ $U(\mathcal{L}\mathfrak{g})$ is a much simpler alg. than $U_q(\mathcal{L}\mathfrak{g})$.

○ $\mathfrak{g} = \mathfrak{g} \otimes 1 \subseteq \mathfrak{g} \otimes \mathbb{C}[t^{\pm 1}] = \mathcal{L}\mathfrak{g}$: Lie subalg., $U_q(\mathfrak{g}) \subseteq U_q(\mathcal{L}\mathfrak{g})$: subalg.,

and $V \cong_{U_q(\mathfrak{g})} \bigoplus_{\lambda} V_q(\lambda)^{\oplus m_{\lambda}} \Leftrightarrow \bar{V} \cong_{\mathfrak{g}} \bigoplus_{\lambda} V(\lambda)^{\oplus m_{\lambda}}$,

that is, taking the classical limit does not lose any information of the $U_q(\mathfrak{g})$ -module structure (in particular, $\text{ch } V = \text{ch } \bar{V}$)

bad point

f.d. $\mathcal{L}\mathfrak{g}$ -modules are not semisimple, and even V is simple, \bar{V} is

rarely simple \Rightarrow need to treat a nonsimple, indec. $\mathcal{L}\mathfrak{g}$ -mod.

classical limits of f.d. $U_q(\mathcal{L}\mathfrak{g})$ -modules

V : f.d. $U_q(\mathcal{L}\mathfrak{g})$ -mod. with some conditions (e.g. almost all simples)

$\lim_{q \rightarrow 1} \bar{V}: U(\mathcal{L}\mathfrak{g})$ -mod. = $\mathcal{L}\mathfrak{g}$ -mod.

Goal Study the structure of V by observing \bar{V} instead.

good point ○ $U(\mathcal{L}\mathfrak{g})$ is a much simpler alg. than $U_q(\mathcal{L}\mathfrak{g})$.

○ $\mathfrak{g} = \mathfrak{g} \otimes 1 \subseteq \mathfrak{g} \otimes \mathbb{C}[t^{\pm 1}] = \mathcal{L}\mathfrak{g}$: Lie subalg., $U_q(\mathfrak{g}) \subseteq U_q(\mathcal{L}\mathfrak{g})$: subalg.,

and $V \cong_{U_q(\mathfrak{g})} \bigoplus_{\lambda} V_q(\lambda)^{\oplus m_{\lambda}} \Leftrightarrow \bar{V} \cong_{\mathfrak{g}} \bigoplus_{\lambda} V(\lambda)^{\oplus m_{\lambda}}$,

that is, taking the classical limit does not lose any information of the $U_q(\mathfrak{g})$ -module structure (in particular, $\text{ch } V = \text{ch } \bar{V}$)

bad point

f.d. $\mathcal{L}\mathfrak{g}$ -modules are not semisimple, and even V is simple, \bar{V} is

rarely simple \Rightarrow need to treat a nonsimple, indec. $\mathcal{L}\mathfrak{g}$ -mod.

classical limits of f.d. $U_q(\mathcal{L}\mathfrak{g})$ -modules

V : f.d. $U_q(\mathcal{L}\mathfrak{g})$ -mod. with some conditions (e.g. almost all simples)

$\lim_{q \rightarrow 1} \bar{V}$: $U(\mathcal{L}\mathfrak{g})$ -mod. = $\mathcal{L}\mathfrak{g}$ -mod.

Goal Study the structure of V by observing \bar{V} instead.

good point ○ $U(\mathcal{L}\mathfrak{g})$ is a much simpler alg. than $U_q(\mathcal{L}\mathfrak{g})$.

○ $\mathfrak{g} = \mathfrak{g} \otimes 1 \subseteq \mathfrak{g} \otimes \mathbb{C}[t^{\pm 1}] = \mathcal{L}\mathfrak{g}$: Lie subalg., $U_q(\mathfrak{g}) \subseteq U_q(\mathcal{L}\mathfrak{g})$: subalg.,

and $V \cong_{U_q(\mathfrak{g})} \bigoplus_{\lambda} V_q(\lambda)^{\oplus m_{\lambda}} \Leftrightarrow \bar{V} \cong_{\mathfrak{g}} \bigoplus_{\lambda} V(\lambda)^{\oplus m_{\lambda}}$,

that is, taking the classical limit does not lose any information of the $U_q(\mathfrak{g})$ -module structure (in particular, $\text{ch } V = \text{ch } \bar{V}$)

bad point

f.d. $\mathcal{L}\mathfrak{g}$ -modules are not semisimple, and even V is simple, \bar{V} is

rarely simple \Rightarrow need to treat a nonsimple, indec. $\mathcal{L}\mathfrak{g}$ -mod.

(1) classical limits of simple modules

Strategy

Given V : simple $U_q(\mathcal{L}\mathfrak{g})$ -mod. $\xrightarrow{\lim_{q \rightarrow 1}} \bar{V}$: $U(\mathcal{L}\mathfrak{g})$ -mod. (= $\mathcal{L}\mathfrak{g}$ -mod.)

Basically, the study of \bar{V} is divided into the following two steps:

- 1 \bar{V} is generated by a distinguished vector called an ℓ -highest weight vector (ℓ -h.w.v.) v . The first step is to find a “good” defining relations of v , i.e., find a “good” subset $S \subset U(\mathcal{L}\mathfrak{g})$ s.t. $\text{Ann}(v) = U(\mathcal{L}\mathfrak{g})S$ ($\Leftrightarrow U(\mathcal{L}\mathfrak{g})/U(\mathcal{L}\mathfrak{g})S \xrightarrow{\sim} \bar{V}$).
- 2 Using $U(\mathcal{L}\mathfrak{g})/U(\mathcal{L}\mathfrak{g})S \cong \bar{V}$, determine its character, simple \mathfrak{g} -module dec., etc.

As a consequence, we obtain the information on the structure of V such as character, simple $U_q(\mathfrak{g})$ -mod. dec., etc.

(1) classical limits of simple modules

Strategy

Given V : simple $U_q(\mathcal{L}\mathfrak{g})$ -mod. $\xrightarrow{\lim_{q \rightarrow 1}} \bar{V}$: $U(\mathcal{L}\mathfrak{g})$ -mod. (= $\mathcal{L}\mathfrak{g}$ -mod.)

Basically, the study of \bar{V} is divided into the following two steps:

- ① \bar{V} is generated by a distinguished vector called an **ℓ -highest weight vector** (ℓ -h.w.v.) v . The first step is to find a “good” defining relations of v , i.e., find a “good” subset $S \subset U(\mathcal{L}\mathfrak{g})$ s.t. $\text{Ann}(v) = U(\mathcal{L}\mathfrak{g})S$ ($\Leftrightarrow U(\mathcal{L}\mathfrak{g})/U(\mathcal{L}\mathfrak{g})S \xrightarrow{\sim} \bar{V}$).
- ② Using $U(\mathcal{L}\mathfrak{g})/U(\mathcal{L}\mathfrak{g})S \cong \bar{V}$, determine its character, simple \mathfrak{g} -module dec., etc.

As a consequence, we obtain the information on the structure of V such as character, simple $U_q(\mathfrak{g})$ -mod. dec., etc.

(1) classical limits of simple modules

Strategy

Given V : simple $U_q(\mathcal{L}\mathfrak{g})$ -mod. $\xrightarrow{\lim_{q \rightarrow 1}} \bar{V}$: $U(\mathcal{L}\mathfrak{g})$ -mod. (= $\mathcal{L}\mathfrak{g}$ -mod.)

Basically, the study of \bar{V} is divided into the following two steps:

- ① \bar{V} is generated by a distinguished vector called an **ℓ -highest weight vector** (ℓ -h.w.v.) v . The first step is to find a “good” defining relations of v , i.e., find a “good” subset $S \subset U(\mathcal{L}\mathfrak{g})$ s.t. $\text{Ann}(v) = U(\mathcal{L}\mathfrak{g})S$ ($\Leftrightarrow U(\mathcal{L}\mathfrak{g})/U(\mathcal{L}\mathfrak{g})S \xrightarrow{\sim} \bar{V}$).
- ② Using $U(\mathcal{L}\mathfrak{g})/U(\mathcal{L}\mathfrak{g})S \cong \bar{V}$, determine its character, simple \mathfrak{g} -module dec., etc.

As a consequence, we obtain the information on the structure of V such as character, simple $U_q(\mathfrak{g})$ -mod. dec., etc.

(1) classical limits of simple modules

Strategy

Given V : simple $U_q(\mathcal{L}\mathfrak{g})$ -mod. $\xrightarrow{\lim_{q \rightarrow 1}} \bar{V}$: $U(\mathcal{L}\mathfrak{g})$ -mod. (= $\mathcal{L}\mathfrak{g}$ -mod.)

Basically, the study of \bar{V} is divided into the following two steps:

- ① \bar{V} is generated by a distinguished vector called an **ℓ -highest weight vector** (ℓ -h.w.v.) v . The first step is to find a “good” defining relations of v , i.e., find a “good” subset $S \subset U(\mathcal{L}\mathfrak{g})$ s.t. $\text{Ann}(v) = U(\mathcal{L}\mathfrak{g})S$ ($\Leftrightarrow U(\mathcal{L}\mathfrak{g})/U(\mathcal{L}\mathfrak{g})S \xrightarrow{\sim} \bar{V}$).
- ② Using $U(\mathcal{L}\mathfrak{g})/U(\mathcal{L}\mathfrak{g})S \cong \bar{V}$, determine its character, simple \mathfrak{g} -module dec., etc.

As a consequence, we obtain the information on the structure of V such as character, simple $U_q(\mathfrak{g})$ -mod. dec., etc.

Notation and Definition

Recall

$\{\text{f.d. simple } U_q(\mathcal{L}\mathfrak{g})\text{-mod.}\} \xrightarrow{1-1} \{(\pi_1(u), \dots, \pi_n(u)) \mid \pi_k(0) = 1\} =: \mathcal{D}$

For $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}_{\geq 0}^n$, $\mathcal{D}^\lambda := \{(\pi_1, \dots, \pi_n) \mid \deg \pi_k = \lambda_k\} \subseteq \mathcal{D}$

Fact If $\pi \in \mathcal{D}^\lambda$, $L_q(\pi) \cong_{U_q(\mathfrak{g})} V_q(\lambda) \oplus (\text{smaller simple } U_q(\mathfrak{g})\text{-mod.})$.

Definition

For $\lambda \in \mathbb{Z}_{\geq 0}^n$, $L_q(\pi)$ is a **minimal affinization** with weight λ

$\stackrel{\text{def}}{\Leftrightarrow}$ smallest dimensional among $\{L_q(\pi) \mid \pi \in \mathcal{D}^\lambda\}$.

The case $\lambda = \ell e_i = (0, \dots, 0, \ell, 0, \dots, 0)$ ($1 \leq i \leq n, \ell \in \mathbb{Z}_{>0}$) is particularly important in integrable system. These special min. affiniz. are called **Kirillov-Reshetikhin (KR) modules** with weight ℓe_i

Notation and Definition

Recall

$\{\text{f.d. simple } U_q(\mathcal{L}\mathfrak{g})\text{-mod.}\} \xrightarrow{1-1} \{(\pi_1(u), \dots, \pi_n(u)) \mid \pi_k(0) = 1\} =: \mathcal{D}$

For $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}_{\geq 0}^n$, $\mathcal{D}^\lambda := \{(\pi_1, \dots, \pi_n) \mid \deg \pi_k = \lambda_k\} \subseteq \mathcal{D}$

Fact If $\pi \in \mathcal{D}^\lambda$, $L_q(\pi) \cong_{U_q(\mathfrak{g})} V_q(\lambda) \oplus (\text{smaller simple } U_q(\mathfrak{g})\text{-mod.})$.

Definition

For $\lambda \in \mathbb{Z}_{\geq 0}^n$, $L_q(\pi)$ is a **minimal affinization** with weight λ

$\stackrel{\text{def}}{\Leftrightarrow}$ smallest dimensional among $\{L_q(\pi) \mid \pi \in \mathcal{D}^\lambda\}$.

The case $\lambda = \ell e_i = (0, \dots, 0, \ell, 0, \dots, 0)$ ($1 \leq i \leq n, \ell \in \mathbb{Z}_{>0}$) is particularly important in integrable system. These special min. affiniz. are called **Kirillov-Reshetikhin (KR) modules** with weight ℓe_i

Notation and Definition

Recall

$\{\text{f.d. simple } U_q(\mathcal{L}\mathfrak{g})\text{-mod.}\} \xrightarrow{1-1} \{(\pi_1(u), \dots, \pi_n(u)) \mid \pi_k(0) = 1\} =: \mathcal{D}$

For $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}_{\geq 0}^n$, $\mathcal{D}^\lambda := \{(\pi_1, \dots, \pi_n) \mid \deg \pi_k = \lambda_k\} \subseteq \mathcal{D}$

Fact If $\pi \in \mathcal{D}^\lambda$, $L_q(\pi) \cong_{U_q(\mathfrak{g})} V_q(\lambda) \oplus (\text{smaller simple } U_q(\mathfrak{g})\text{-mod.})$.

Definition

For $\lambda \in \mathbb{Z}_{\geq 0}^n$, $L_q(\pi)$ is a **minimal affinization** with weight λ

$\stackrel{\text{def}}{\Leftrightarrow}$ smallest dimensional among $\{L_q(\pi) \mid \pi \in \mathcal{D}^\lambda\}$.

The case $\lambda = \ell e_i = (0, \dots, 0, \ell, 0, \dots, 0)$ ($1 \leq i \leq n, \ell \in \mathbb{Z}_{>0}$) is particularly important in integrable system. These special min. affiniz. are called **Kirillov-Reshetikhin (KR) modules** with weight ℓe_i

classical limits of Kirillov-Reshetikhin module

Let $\{W^{i,\ell}(a) \mid a \in \mathbb{C}(q)^\times\}$ denote the KR modules with weight ℓe_i .

Study of KR mod. in integrable system

\rightsquigarrow conjectural formula for simple $U_q(\mathfrak{g})$ -mod. dec.:

$$W^{i,\ell}(a) \cong_{U_q(\mathfrak{g})} \bigoplus V_q(\lambda)^{\oplus m_{i,\ell}(\lambda)} \quad (\text{KR conjecture})$$

Theorem (Chari, 01)

Assume that \mathfrak{g} is of type A_n, B_n, C_n or D_n , and $v \in W^{i,\ell}(a)$ is ℓ -h.w.v.

(i) $\text{Ann}(v) = U(\mathcal{L}\mathfrak{g}) S$ with a certain subset S of elements in $U(\mathcal{L}\mathfrak{g})$

$$(S = \mathfrak{n}_+[t] \cup (\mathfrak{h} \otimes (t - a(1))) \cup \{f_i^{\ell+1}, f_j (j \neq i), f_i \otimes t\})$$

(ii) KR conj. holds.

Recall Simple Lie algebras are classified into types

A_n (\mathfrak{sl}_{n+1}), B_n (\mathfrak{so}_{2n+1}), C_n (\mathfrak{sp}_{2n}), D_n (\mathfrak{so}_{2n}), E_6, E_7, E_8, F_4, G_2 .

classical limits of Kirillov-Reshetikhin module

Let $\{W^{i,\ell}(a) \mid a \in \mathbb{C}(q)^\times\}$ denote the KR modules with weight ℓe_i .

Study of KR mod. in integrable system

\rightsquigarrow conjectural formula for simple $U_q(\mathfrak{g})$ -mod. dec.:

$$W^{i,\ell}(a) \cong_{U_q(\mathfrak{g})} \bigoplus V_q(\lambda)^{\oplus m_{i,\ell}(\lambda)} \quad (\text{KR conjecture})$$

Theorem (Chari, 01)

Assume that \mathfrak{g} is of type A_n, B_n, C_n or D_n , and $v \in W^{i,\ell}(a)$ is ℓ -h.w.v.

(i) $\text{Ann}(v) = U(\mathcal{L}\mathfrak{g}) S$ with a certain subset S of elements in $U(\mathcal{L}\mathfrak{g})$

$$(S = \mathfrak{n}_+[t] \cup (\mathfrak{h} \otimes (t - a(1))) \cup \{f_i^{\ell+1}, f_j (j \neq i), f_i \otimes t\})$$

(ii) KR conj. holds.

Recall Simple Lie algebras are classified into types

A_n (\mathfrak{sl}_{n+1}), B_n (\mathfrak{so}_{2n+1}), C_n (\mathfrak{sp}_{2n}), D_n (\mathfrak{so}_{2n}), E_6, E_7, E_8, F_4, G_2 .

- (i) $\text{Ann}(v) = U(\mathcal{L}\mathfrak{g})S \Leftrightarrow U(\mathcal{L}\mathfrak{g})/U(\mathcal{L}\mathfrak{g})S \xrightarrow{\sim} \overline{V}$
- (ii) KR conj. holds. $(W^{i,\ell}(a) \cong_{U_q(\mathfrak{g})} \bigoplus V_q(\lambda)^{\oplus m_{i,\ell}(\lambda)})$

sketch of proof

Fact If \mathfrak{g} is of type $ABCD$, $m_{i,\ell}(\lambda) \leq 1$ for any λ .

- $U(\mathcal{L}\mathfrak{g})/U(\mathcal{L}\mathfrak{g})S \rightarrow \overline{V}$ is easy to prove.
- Prove $[U(\mathcal{L}\mathfrak{g})/U(\mathcal{L}\mathfrak{g})S : V(\lambda)] \leq m_{i,\ell}(\lambda)$.
- Prove $[W^{i,\ell}(a) : V_q(\lambda)] \geq m_{i,\ell}(\lambda)$ (hard calculation in $U_q(\mathcal{L}\mathfrak{g})$) \square

It seems difficult to apply this proof to wider class of simple modules!

Rem. KR conj. has been proved in general types using the theory of q -characters.

- (i) $\text{Ann}(v) = U(\mathcal{L}\mathfrak{g})S \ (\Leftrightarrow U(\mathcal{L}\mathfrak{g})/U(\mathcal{L}\mathfrak{g})S \xrightarrow{\sim} \overline{V})$
- (ii) KR conj. holds. $(W^{i,\ell}(a) \cong_{U_q(\mathfrak{g})} \bigoplus V_q(\lambda)^{\oplus m_{i,\ell}(\lambda)})$

sketch of proof

Fact If \mathfrak{g} is of type $ABCD$, $m_{i,\ell}(\lambda) \leq 1$ for any λ .

- $U(\mathcal{L}\mathfrak{g})/U(\mathcal{L}\mathfrak{g})S \rightarrow \overline{V}$ is easy to prove.
- Prove $[U(\mathcal{L}\mathfrak{g})/U(\mathcal{L}\mathfrak{g})S : V(\lambda)] \leq m_{i,\ell}(\lambda)$.
- Prove $[W^{i,\ell}(a) : V_q(\lambda)] \geq m_{i,\ell}(\lambda)$ (hard calculation in $U_q(\mathcal{L}\mathfrak{g})$) \square

It seems difficult to apply this proof to wider class of simple modules!

Rem. KR conj. has been proved in general types using the theory of q -characters.

class. lim. of minimal affinizations

Theorem (N, Li-N, '13-'15)

Assume \mathfrak{g} is of type $ABCD$ or G_2 , and let v be an ℓ -h.w.v. of a minimal affinization W with weight $\lambda \in \mathbb{Z}_{\geq 0}^n$.

Then there exists a “good” subset $S \subseteq U(\mathcal{L}\mathfrak{g})$ s.t. $\text{Ann}(v) = U(\mathcal{L}\mathfrak{g}) S$.

Corollary

By analyzing $U(\mathcal{L}\mathfrak{g})/U(\mathcal{L}\mathfrak{g}) S$, we obtain:

- 1 a Jacobi-Trudi type character formula in type $ABCD$, which expresses the character as a determinant of a matrix (new in CD).
- 2 a polyhedral multiplicity formula in type G_2 , which expresses $[W : V_q(\mu)]$ as # of the lattice pts in a polyhedron.

class. lim. of minimal affinizations

Theorem (N, Li-N, '13-'15)

Assume \mathfrak{g} is of type $ABCD$ or G_2 , and let v be an ℓ -h.w.v. of a minimal affinization W with weight $\lambda \in \mathbb{Z}_{\geq 0}^n$.

Then there exists a “good” subset $S \subseteq U(\mathcal{L}\mathfrak{g})$ s.t. $\text{Ann}(v) = U(\mathcal{L}\mathfrak{g}) S$.

Corollary

By analyzing $U(\mathcal{L}\mathfrak{g})/U(\mathcal{L}\mathfrak{g}) S$, we obtain:

- ① a Jacobi-Trudi type character formula in type $ABCD$, which expresses the character as a determinant of a matrix (**new in CD**).
- ② a polyhedral multiplicity formula in type G_2 , which expresses $[W : V_q(\mu)]$ as $\#$ of the lattice pts in a polyhedron.

Def. $\mathfrak{g}[t] = \mathfrak{g} \otimes \mathbb{C}[t] \subseteq \mathcal{L}\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$: current algebra.

sketch of the poof of $U(\mathcal{L}\mathfrak{g})/U(\mathcal{L}\mathfrak{g})S \xrightarrow{\sim} \overline{W}$

- $U(\mathcal{L}\mathfrak{g})/U(\mathcal{L}\mathfrak{g})S \twoheadrightarrow \overline{W}$ is easy.
- It is enough to show $\exists \text{ surj. } \overline{W} \twoheadrightarrow U(\mathcal{L}\mathfrak{g})/U(\mathcal{L}\mathfrak{g})S$
 $\Leftrightarrow \exists \mathfrak{g}[t]\text{-mod. surj. } \overline{W} \twoheadrightarrow U(\mathfrak{g}[t])/U(\mathfrak{g}[t])S'$, where $S' = U(\mathfrak{g}[t]) \cap S$.

We use the theory of the affine Lie algebra $\widehat{\mathfrak{g}}$ to prove this!

$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K$: affine Lie algebra

$([X \otimes f + aK, Y \otimes g + bK] = [X, Y] \otimes fg + K(X, Y) \text{Res}_{t=0} g \cdot df/dt)$

Rem. $\mathfrak{g}[t]$ is a Lie subalgebra of $\widehat{\mathfrak{g}}$, but $\mathcal{L}\mathfrak{g}$ is not!

Def. $\mathfrak{g}[t] = \mathfrak{g} \otimes \mathbb{C}[t] \subseteq \mathcal{L}\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$: current algebra.

sketch of the poof of $U(\mathcal{L}\mathfrak{g})/U(\mathcal{L}\mathfrak{g})S \xrightarrow{\sim} \overline{W}$

- $U(\mathcal{L}\mathfrak{g})/U(\mathcal{L}\mathfrak{g})S \twoheadrightarrow \overline{W}$ is easy.
- It is enough to show \exists surj. $\overline{W} \twoheadrightarrow U(\mathcal{L}\mathfrak{g})/U(\mathcal{L}\mathfrak{g})S$
 $\Leftrightarrow \exists \mathfrak{g}[t]$ -mod. surj. $\overline{W} \twoheadrightarrow U(\mathfrak{g}[t])/U(\mathfrak{g}[t])S'$, where $S' = U(\mathfrak{g}[t]) \cap S$.

We use the theory of the affine Lie algebra $\widehat{\mathfrak{g}}$ to prove this!

$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K$: affine Lie algebra

$$([X \otimes f + aK, Y \otimes g + bK] = [X, Y] \otimes fg + K(X, Y) \operatorname{Res}_{t=0} g \cdot df/dt)$$

Rem. $\mathfrak{g}[t]$ is a Lie subalgebra of $\widehat{\mathfrak{g}}$, but $\mathcal{L}\mathfrak{g}$ is not!

$\widehat{\mathfrak{g}}$ has a good class of simple modules $\{\widehat{V}(\Lambda) \mid \Lambda \in \mathbb{Z}_{\geq 0}^{n+1}\}$ called integrable highest weight modules.

Definition

A $\mathfrak{g}[t]$ -submodule $U(\mathfrak{g}[t])(v_{w_1\Lambda_1} \otimes \cdots \otimes v_{w_p\Lambda_p}) \subseteq \widehat{V}(\Lambda_1) \otimes \cdots \otimes \widehat{V}(\Lambda_p)$ is called a **generalized Demazure module**, where $\{v_{w\Lambda} \mid w \in \widehat{W}\}$ are distinguished vectors in $\widehat{V}(\Lambda)$ called extremal weight vectors.

Theorem (Joseph, Lakshmibai-Littelmann-Magyar)

By using the representation theory of $\widehat{\mathfrak{g}}$, we can determine defining relations of (a subclass of) generalized Demazure modules.

○ Finally, show that there exist two $\mathfrak{g}[t]$ -mod. surjections:

$$\overline{W} \twoheadrightarrow \varphi_c^* D \twoheadrightarrow U(\mathfrak{g}[t])/U(\mathfrak{g}[t])S' \text{ with } D \text{ a generalized Demazure mod.,}$$

where φ_c is an auto. on $\mathfrak{g}[t]$ defined by $\varphi_c(X \otimes f(t)) = X \otimes f(t+c)$ \square

$\widehat{\mathfrak{g}}$ has a good class of simple modules $\{\widehat{V}(\Lambda) \mid \Lambda \in \mathbb{Z}_{\geq 0}^{n+1}\}$ called integrable highest weight modules.

Definition

A $\mathfrak{g}[t]$ -submodule $U(\mathfrak{g}[t])(v_{w_1\Lambda_1} \otimes \cdots \otimes v_{w_p\Lambda_p}) \subseteq \widehat{V}(\Lambda_1) \otimes \cdots \otimes \widehat{V}(\Lambda_p)$ is called a **generalized Demazure module**, where $\{v_{w\Lambda} \mid w \in \widehat{W}\}$ are distinguished vectors in $\widehat{V}(\Lambda)$ called extremal weight vectors.

Theorem (Joseph, Lakshmibai-Littelmann-Magyar)

By using the representation theory of $\widehat{\mathfrak{g}}$, we can determine defining relations of (a subclass of) generalized Demazure modules.

- Finally, show that there exist two $\mathfrak{g}[t]$ -mod. surjections:

$$\overline{W} \twoheadrightarrow \varphi_c^* D \twoheadrightarrow U(\mathfrak{g}[t])/U(\mathfrak{g}[t]) S' \text{ with } D \text{ a generalized Demazure mod.,}$$

where φ_c is an auto. on $\mathfrak{g}[t]$ defined by $\varphi_c(X \otimes f(t)) = X \otimes f(t+c)$ \square

(2) classical limit of a tensor product

$$V_1, \dots, V_p: \text{f.d. } U_q(\mathcal{L}\mathfrak{g})\text{-mod.} \xrightarrow{\otimes} V_1 \otimes \cdots \otimes V_p: \text{f.d. } U_q(\mathcal{L}\mathfrak{g})\text{-mod.}$$

$$\xrightarrow{\lim_{q \rightarrow 1}} \overline{V_1 \otimes \cdots \otimes V_p}: \mathcal{L}\mathfrak{g}\text{-mod.}$$

$$V_1, \dots, V_p: \text{f.d. } U_q(\mathcal{L}\mathfrak{g})\text{-mod.} \xrightarrow{\lim_{q \rightarrow 1}} \overline{V_1}, \dots, \overline{V_p}: \mathcal{L}\mathfrak{g}\text{-mod.}$$

$$\xrightarrow{\otimes} \overline{V_1} \otimes \cdots \otimes \overline{V_p}: \mathcal{L}\mathfrak{g}\text{-mod.}$$

In many cases, it happens that $\overline{V_1 \otimes \cdots \otimes V_p} \not\cong \overline{V_1} \otimes \cdots \otimes \overline{V_p}$!

i.e. operations of taking tensor prod. and class. lim. are non-commutative!

Q. Can we construct $\overline{V_1 \otimes \cdots \otimes V_p}$ from $\overline{V_1}, \dots, \overline{V_p}$?

We solved this question affirmatively when V_1, \dots, V_p are KR-modules.

(2) classical limit of a tensor product

$$V_1, \dots, V_p: \text{f.d. } U_q(\mathcal{L}\mathfrak{g})\text{-mod.} \xrightarrow{\otimes} V_1 \otimes \dots \otimes V_p: \text{f.d. } U_q(\mathcal{L}\mathfrak{g})\text{-mod.}$$

$$\xrightarrow{\lim_{q \rightarrow 1}} \overline{V_1 \otimes \dots \otimes V_p}: \mathcal{L}\mathfrak{g}\text{-mod.}$$

$$V_1, \dots, V_p: \text{f.d. } U_q(\mathcal{L}\mathfrak{g})\text{-mod.} \xrightarrow{\lim_{q \rightarrow 1}} \overline{V_1}, \dots, \overline{V_p}: \mathcal{L}\mathfrak{g}\text{-mod.}$$

$$\xrightarrow{\otimes} \overline{V_1} \otimes \dots \otimes \overline{V_p}: \mathcal{L}\mathfrak{g}\text{-mod.}$$

In many cases, it happens that $\overline{V_1 \otimes \dots \otimes V_p} \not\cong \overline{V_1} \otimes \dots \otimes \overline{V_p}$!

i.e. operations of taking tensor prod. and class. lim. are non-commutative!

Q. Can we construct $\overline{V_1 \otimes \dots \otimes V_p}$ from $\overline{V_1}, \dots, \overline{V_p}$?

We solved this question affirmatively when V_1, \dots, V_p are KR-modules.

(2) classical limit of a tensor product

$$V_1, \dots, V_p: \text{f.d. } U_q(\mathcal{L}\mathfrak{g})\text{-mod.} \xrightarrow{\otimes} V_1 \otimes \dots \otimes V_p: \text{f.d. } U_q(\mathcal{L}\mathfrak{g})\text{-mod.}$$

$$\xrightarrow{\lim_{q \rightarrow 1}} \overline{V_1 \otimes \dots \otimes V_p}: \mathcal{L}\mathfrak{g}\text{-mod.}$$

$$V_1, \dots, V_p: \text{f.d. } U_q(\mathcal{L}\mathfrak{g})\text{-mod.} \xrightarrow{\lim_{q \rightarrow 1}} \overline{V_1}, \dots, \overline{V_p}: \mathcal{L}\mathfrak{g}\text{-mod.}$$

$$\xrightarrow{\otimes} \overline{V_1} \otimes \dots \otimes \overline{V_p}: \mathcal{L}\mathfrak{g}\text{-mod.}$$

In many cases, it happens that $\overline{V_1 \otimes \dots \otimes V_p} \not\cong \overline{V_1} \otimes \dots \otimes \overline{V_p}$!

i.e. operations of taking tensor prod. and class. lim. are non-commutative!

Q. Can we construct $\overline{V_1 \otimes \dots \otimes V_p}$ from $\overline{V_1}, \dots, \overline{V_p}$?

We solved this question affirmatively when V_1, \dots, V_p are KR-modules.

(2) classical limit of a tensor product

$$V_1, \dots, V_p: \text{f.d. } U_q(\mathcal{L}\mathfrak{g})\text{-mod.} \xrightarrow{\otimes} V_1 \otimes \dots \otimes V_p: \text{f.d. } U_q(\mathcal{L}\mathfrak{g})\text{-mod.}$$

$$\xrightarrow{\lim_{q \rightarrow 1}} \overline{V_1 \otimes \dots \otimes V_p}: \mathcal{L}\mathfrak{g}\text{-mod.}$$

$$V_1, \dots, V_p: \text{f.d. } U_q(\mathcal{L}\mathfrak{g})\text{-mod.} \xrightarrow{\lim_{q \rightarrow 1}} \overline{V_1}, \dots, \overline{V_p}: \mathcal{L}\mathfrak{g}\text{-mod.}$$

$$\xrightarrow{\otimes} \overline{V_1} \otimes \dots \otimes \overline{V_p}: \mathcal{L}\mathfrak{g}\text{-mod.}$$

In many cases, it happens that $\overline{V_1 \otimes \dots \otimes V_p} \not\cong \overline{V_1} \otimes \dots \otimes \overline{V_p}$!

i.e. operations of taking tensor prod. and class. lim. are non-commutative!

Q. Can we construct $\overline{V_1 \otimes \dots \otimes V_p}$ from $\overline{V_1}, \dots, \overline{V_p}$?

We solved this question affirmatively when V_1, \dots, V_p are KR-modules.

Theorem (N)

Assume the tensor product of KR modules $W^{i_1, \ell_1}(a_1) \otimes \cdots \otimes W^{i_p, \ell_p}(a_p)$ has a classical limit (\exists sufficient conditions).

(i) If $a_1(1) = \cdots = a_p(1) =: c \in \mathbb{C}^\times$, then

$$\overline{W^{i_1, \ell_1}(a_1) \otimes \cdots \otimes W^{i_p, \ell_p}(a_p)} \cong \varphi_c^*(\overline{W^{i_1, \ell_1}(a_1)} * \cdots * \overline{W^{i_p, \ell_p}(a_p)}),$$

where $*$ is the fusion product explained below.

(ii) In the general case, we have

$$\overline{W^{i_1, \ell_1}(a_1) \otimes \cdots \otimes W^{i_p, \ell_p}(a_p)} \cong \bigotimes_{c \in \mathbb{C}^\times} \varphi_c^* \left(\underset{k; a_k(1)=c}{*} \overline{W^{i_k, \ell_k}(a_k)} \right).$$

fusion prod. For $\mathfrak{g}[t]$ -modules V_1, \dots, V_p (with some conditions), their fusion prod. $V_1 * \cdots * V_p$ is defined as a graded analog of tensor product, and it was defined by Feigin-Loktev as a “graded analog of the conformal coinvariants” in conformal field theory.