Existence of Kirillov–Reshetikhin crystals for the near adjoint nodes in exceptional types

Katsuyuki Naoi (joint work with Travis Scrimshaw)

Tokyo University of Agriculture and Technology

Representation Theory of Algebraic Groups and Quantum Groups
– in honor of Professor Ariki's 60th birthday –

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Introduction

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(e.g. $\mathfrak{g} = \mathfrak{g}_0 \otimes \mathbb{Q}[t, t^{-1}] \oplus \mathbb{Q}K$, $\mathfrak{g}_0$: simple Lie alg.)

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Conjecture (Hatayama,Kuniba,Okado,Takgagi,Yamada/Tsuboi, 99-01)

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KR mod. $W^{r,\ell}$: a family of f.d. simple $U_q'(\mathfrak{g})$-mod ($r \in I \setminus \{0\}$, $\ell \in \mathbb{Z}_{>0}$)

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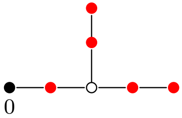
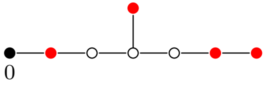


$E_7^{(1)}$:



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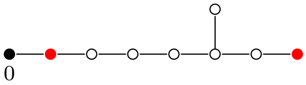

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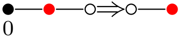

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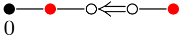
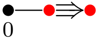
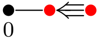


$G_2^{(1)}$:



$D_4^{(3)}$:



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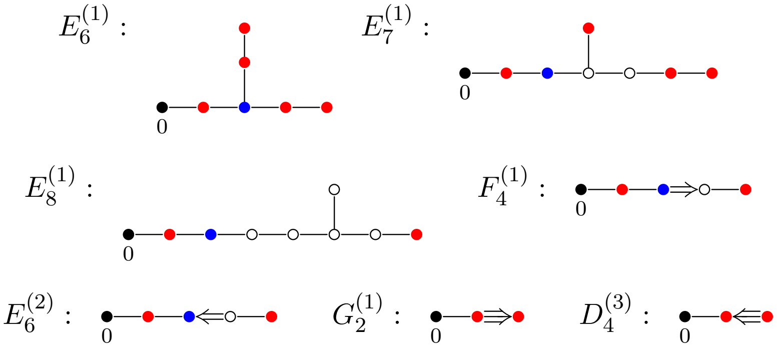

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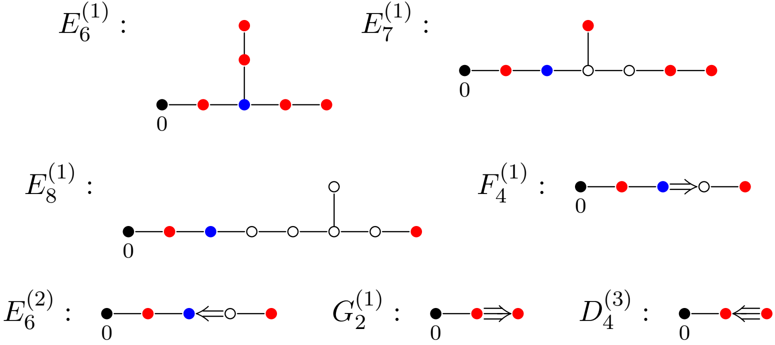
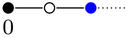


Rem. In all types, the local diagrams are the same:



Plan

1. Basic notions
   - Crystal bases and pseudobases
   - KR modules
   - Prepolarization
2. Criterion for the existence of a crystal pseudobase

   by [Kang–Kashiwara–Misra–Miwa–Nakashima–Nakayashiki, 92]
3. Proof
4. Future work

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crystal base and pseudobase

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$U_q'(\mathfrak{g})$: quantum affine algebra without degree operator q^d
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M: integrable $U'_q(\mathfrak{g})$-module,

$e_i, f_i \curvearrowright M \quad (i \in I) \overset{\text{“twist”}}{\rightsquigarrow} \tilde{e}_i, \tilde{f}_i \curvearrowright M \ (i \in I)$: **Kashiwara operators**

Definition

(1) A pair (L, B) is called a **crystal base** if

(a) L: A-lattice of M ($A := \{f/g \mid g(0) \neq 0\} \subseteq \mathbb{Q}(q)$: local subring),

(b) $B \subseteq L/qL$: a $\mathbb{Q}$-basis,

(c) $L = \bigoplus_\lambda L_\lambda$, $B = \bigsqcup_\lambda B_\lambda$ (i.e. compatible with weight dec.),

(d) $\tilde{e}_i L, \tilde{f}_i L \subseteq L$ ($\Rightarrow \tilde{e}_i, \tilde{f}_i \curvearrowright L/qL$),

(e) $\tilde{e}_i b, \tilde{f}_i b \in B \sqcup \{0\}$ for $b \in B$,

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(2) A pair (L, B) is called a **crystal pseudobase** if (a), (c)–(f) and

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Rem. In the same way with crystal bases, from a crystal pseudobase we can construct a **crystal graph** (I-colored oriented graph)

$\rightsquigarrow$ combinatorial formulas for tensor products, branching rules, etc.

Kirillov-Reshetikhin (KR) modules

$$U_q'(\mathfrak{g}) \supseteq U_q(\mathfrak{g}_0) := \mathbb{Q}(q)\langle e_i, f_i, q^{h_i} \mid i \in I_0 := I \setminus \{0\}\rangle$$

P_0: weight lattice of $\mathfrak{g}_0$, P_0^+: set of dominant integral weights of $\mathfrak{g}_0$,

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<u>Fact</u> {isom. classes of simple $U_q(\mathfrak{g}_0)$-modules} $\overset{1:1}{\longleftrightarrow} P_0^+$

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$W^r_{q^k} = W^r$ as vector sp., and denoting by ρ the new action, we have

$\rho(e_i)v = q^{\delta_{0i}k} e_i v, \ \ \rho(f_i)v = q^{-\delta_{0i}k} f_i v, \ \ \rho(q^{h_i})v = q^{h_i} v.$

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For $r \in I_0$ and $\ell \in \mathbb{Z}_{>0}$, consider a nontrivial $U_q'(\mathfrak{g})$-module hom.

$$W^r_{q^{\ell-1}} \otimes W^r_{q^{\ell-3}} \otimes \cdots \otimes W^r_{q^{-\ell+1}} \xrightarrow{R} W^r_{q^{-\ell+1}} \otimes \cdots \otimes W^r_{q^{\ell-3}} \otimes W^r_{q^{\ell-1}}.$$

Definition

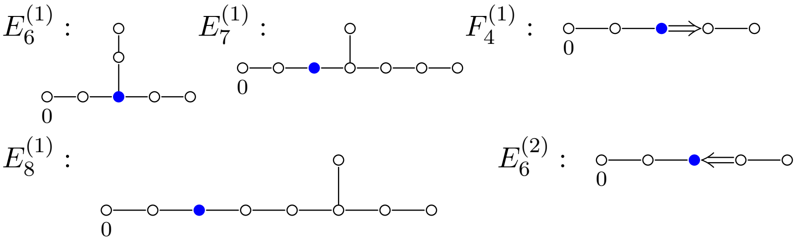
$W^{r,\ell} := \operatorname{Im} R$: **Kirillov-Reshetikhin** (KR) **modules**

Note $W^{r,1} = W^r$.

Main Theorem

Theorem (N–Scrimshaw)

If $\mathfrak{g}$ is either of type $E_{6,7,8}^{(1)}$, $F_4^{(1)}$ or $E_6^{(2)}$ and r is near adjoint, then the KR module $W^{r,\ell}$ has a crystal pseudobase for every ℓ.



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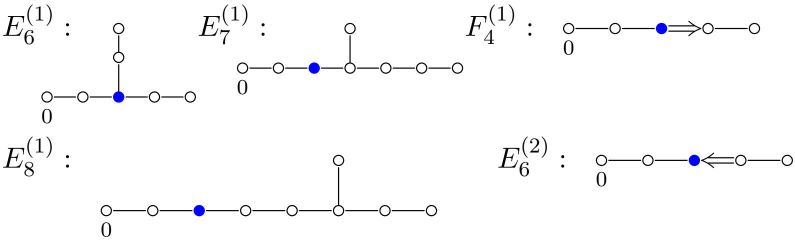

In the proof, we use a **criterion** introduced by [KKMMNN]:

($^\exists$crystal pseudobase) $\Leftarrow$ statements on a **prepolarization** (,)

prepolarization

Define an anti-involution Ψ of $U_q'(\mathfrak{g})$ by

$$\Psi(e_i) = q_i^{-1} q_i^{-h_i} f_i, \quad \Psi(f_i) = q_i^{-1} q_i^{h_i} e_i, \quad \Psi(q^{h_i}) = q^{h_i},$$

where $q_i = q^{c_i}$ with a certain positive integer c_i.

Definition

Let M be a $U_q'(\mathfrak{g})$-module, and $(\ ,\ )$ a $\mathbb{Q}(q)$-bilinear form on M.
We say $(\ ,\ )$ is a **prepolarization** on M if it is symmetric
and satisfies $(xu, v) = (u, \Psi(x)v)$ for $x \in U_q'(\mathfrak{g})$ and $u, v \in M$.

In this talk, we often use the notation $||u||^2 = (u, u)$.

Proposition

$W^{r,\ell}$ has a prepolarization $(\ ,\ )$.

Construction of this prepolarization

Recall $W^{r,\ell} := \operatorname{Im} R$, where

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<u>Fact</u> W^r has a prepolarization $(\ , \ )$.

$\rightsquigarrow$ natural pairing $(\ , \ )$ between $W^r_{q^k}$ and $W^r_{q^{-k}}$ for any $k \in \mathbb{Z}$.

$\rightsquigarrow$ $(u_1 \otimes \cdots \otimes u_\ell, v_1 \otimes \cdots \otimes v_\ell)' = (u_1, v_1) \cdots (u_\ell, v_\ell)$ defines a pairing between $W^r_{q^{\ell-1}} \otimes \cdots \otimes W^r_{q^{-\ell+1}}$ and $W^r_{q^{-\ell+1}} \otimes \cdots \otimes W^r_{q^{\ell-1}}$.

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Then $\big(R(u), R(v)\big) := \big(u, R(v)\big)'$ for $u, v \in W^r_{q^{\ell-1}} \otimes \cdots \otimes W^r_{q^{-\ell+1}}$

Criterion for the existence of crystal pseudobase

Theorem (KKMMNN)

Let M be a f.d. $U_q'(\mathfrak{g})$-module, and assume that

(1) M has a prepolarization $(\ ,\ )$,

(2) $^\exists$"suitable $\mathbb{Z}$-form" $M_\mathbb{Z}$ in M,

(3) there exists a set of vectors $S = \{u_1, \ldots, u_m\} \subseteq M_\mathbb{Z}$ s.t.

- (i) $M \cong_{U_q(\mathfrak{g}_0)} \bigoplus_{k=1}^m V_0\big(\mathrm{wt}(u_k)\big)$,
- (ii) $(u_k, u_j) \in \delta_{kj} + qA \quad ({}^\forall k, j)$ **(almost orthonomality)**
- (iii) $||e_i u_k||^2 \in q_i^{-2\langle h_i, \mathrm{wt}(u_k)\rangle - 1} A \quad ({}^\forall i \in I_0,\ {}^\forall k)$.

Then, setting

$L := \{u \in M \mid ||u||^2 \in A\}$, $B := \{b \in (M_\mathbb{Z} \cap L)/(M_\mathbb{Z} \cap qL) \mid ||b||^2 = 1\}$,

(L, B) is a crystal pseudobase of M.

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- (i) $M \cong_{U_q(\mathfrak{g}_0)} \bigoplus_{k=1}^m V_0\big(\mathrm{wt}(u_k)\big)$,
- (ii) $(u_k, u_j) \in \delta_{kj} + qA \quad ({}^\forall k, j)$ **(almost orthonomality)**
- (iii) $||e_i u_k||^2 \in q_i^{-2\langle h_i, \mathrm{wt}(u_k)\rangle - 1} A \quad ({}^\forall i \in I_0,\ {}^\forall k)$.

Then, setting

$L := \{u \in M \mid ||u||^2 \in A\}$, $B := \{b \in (M_\mathbb{Z} \cap L)/(M_\mathbb{Z} \cap qL) \mid ||b||^2 = 1\}$,

(L, B) is a crystal pseudobase of M.

<u>Note</u> (ii) $\Rightarrow b_k := \overline{u_k} \in B$, (iii) $\Rightarrow \tilde{e}_i b_k = 0$ $(i \in I_0)$.

So (i)–(iii) imply that there exist enough $U_q(\mathfrak{g}_0)$-h.w. elements in B.

what we need to do in KR module case

(1) M has a prepolarization $(\ ,\ )$,

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(1) and (2) are known to hold for all the KR modules $W^{r,\ell}$.

Hence what we have to do is the following:

(a) Find a suitable set $S_\ell = \{u_1, \dots, u_m\} \subseteq W^{r,\ell}$,

(b) Check that these vectors satisfy (i), (ii) and (iii).

Proof of the main theorem

Theorem (N–Scrimshaw)

If $\mathfrak{g}$ is either of type $E_{6,7,8}^{(1)}$, $F_4^{(1)}$ or $E_6^{(2)}$ and r is near adjoint, then the KR module $W^{r,\ell}$ has a crystal pseudobase for every ℓ.

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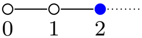
In the sequel, assume that $\mathfrak{g}$ is either of type $E_{6,7,8}^{(1)}$, $F_4^{(1)}$ or $E_6^{(2)}$, and the nodes are labelled as 

(i.e., 2: near adjoint node)

We have to

(a) Find a suitable set $S_\ell = \{u_1, \ldots, u_m\} \subseteq W^{2,\ell}$,

(b) Check that these vectors satisfy (i) $W^{2,\ell} \cong \bigoplus V_0\big(\mathrm{wt}(u_k)\big)$,
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Construction of the set of vectors S_ℓ in the criterion

Notation $[m] = (q^m - q^{-m})/(q - q^{-1})$, $[m]! = [m] \cdots [1]$,

Set $e_i^{(m)} = e_i^m/[m]!$ for $i \in I$ (q-devided power),

$w_\ell \in W^{2,\ell}_{\ell\varpi_2}$: a highest weight vector s.t. $||w_\ell||^2 = 1$,

For a seq. $i_1, i_2, \ldots, i_p$ of elements of I, $e^{(m)}_{i_1,i_2,\ldots,i_p} := e^{(m)}_{i_1} e^{(m)}_{i_2} \cdots e^{(m)}_{i_p}$.

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For $\boldsymbol{a} = (a_1, \ldots, a_6) \in \mathbb{Z}^6_{\geq 0}$, $e^{\boldsymbol{a}} := e_0^{(a_6)} e_1^{(a_5)} e_2^{(a_4)} E_\beta^{(a_3)} E_\alpha^{(a_2)} e_{1,0}^{(a_1)}$,

where $E_\alpha^{(a)}$, $E_\beta^{(a)}$ are some prod. of $e_i^{(a)}$'s $\Leftarrow$ defined in the next slide

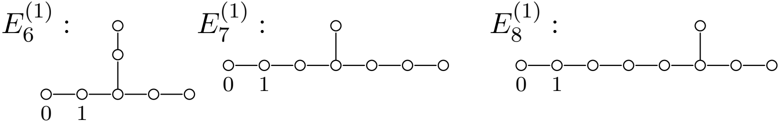
Definition

For $\ell \in \mathbb{Z}_{>0}$, define a subset $S_\ell \subseteq W^{2,\ell}$ by

$$S_\ell := \{e^{\boldsymbol{a}} w_\ell \mid a_6 \leq a_5 \leq a_4 \leq a_3 \leq a_2,\ a_2 + a_3 + a_4 - a_5 \leq a_1 \leq a_4 + \ell\}.$$

Assume that $\mathfrak{g}$ is either of type $E_{6,7,8}^{(1)}$.

α: the highest root of $I \setminus \{0, 1\}$,



Assume that $\mathfrak{g}$ is either of type $E^{(1)}_{6,7,8}$.

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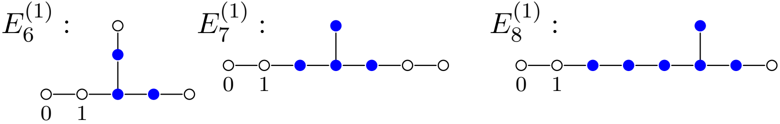

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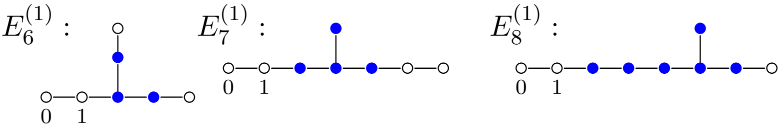

Set $E_\alpha^{(a)} := e_{i_1,\dots,i_r,2}^{(a)}$, $E_\beta^{(a)} := e_{j_1,\dots,j_s,2}^{(a)}$ for $m \in \mathbb{Z}_{\geq 0}$, where

$i_1,\dots,i_r$: a (nonredundant) seq. of el. of I s.t. $s_{i_1}\cdots s_{i_r}(\alpha_2) = \alpha$,

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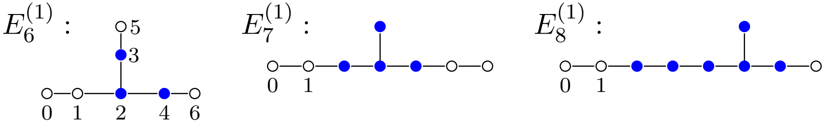

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Ex. ($\mathfrak{g} = E_6^{(1)}$)

$$\alpha = \alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6 = s_6 s_5 s_4 s_3(\alpha_2) \Rightarrow E_\alpha^{(m)} = e^{(m)}_{6,5,4,3,2},$$

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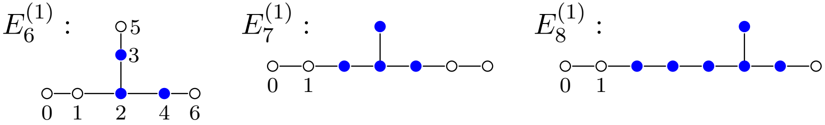

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In types $F_4^{(1)}$, $E_6^{(2)}$, one defines $E^{(m)}_\alpha$, $E^{(m)}_\beta$ in a slmilar way.

How to find the set $S_\ell = \{e^{\boldsymbol{a}} w_\ell \mid \cdots\} \subseteq W^{2,\ell}$?

$^{\exists}$combin. formula for dec. $W^{r,\ell} \cong_{U_q(\mathfrak{g}_0)} \bigoplus_\lambda V_0(\lambda)$ **(fermionic formula)**

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Observation In previous works (e.g., [Okado-Schilling]):

vectors $\{u_k\}$ are written in the forms $e_{i_1}^{(a_1)} \cdots e_{i_p}^{(a_p)} w_\ell$ $(a_1, \ldots, a_p \in \mathbb{Z}_{\geq 0})$.

Here $s_{i_1} \cdots s_{i_p}$: red. exp. of an el. $w \in W_{\mathrm{aff}}$ s.t. $w(\varpi_r + \Lambda_0) \in P^+$

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By assuming this also holds in our cases,

the set of vectors $S_\ell = \{e_0^{(a_6)} e_1^{(a_5)} e_2^{(a_4)} E_\beta^{(a_3)} E_\alpha^{(a_2)} e_{1,0}^{(a_1)} w_\ell \mid \cdots\}$ was found.

For the proof of the Main Theorem, we have to check that

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idea

Use the theory of **global bases** in **extremal weight modules**!

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affine weight $\mu \in P \rightsquigarrow$ extremal weight module $V(\mu)$ [Kashiwara, 94]

($U_q(\mathfrak{g})$-mod. with a generator v_μ of weight μ and certain defining rel.)

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Theorem (Beck-Nakajima, 04)

$V(\mu)$ has a prepolarization $(\ ,\ )$, and we have $(G(b), G(b')) \in \delta_{bb'} + qA$.

We will give a sketch of the proof for the almost orthonormality:
$(e^{\boldsymbol{a}} w_\ell, e^{\boldsymbol{a}'} w_\ell) \in \delta_{\boldsymbol{a}\boldsymbol{a}'} + qA \quad (e^{\boldsymbol{a}} w_\ell, e^{\boldsymbol{a}'} w_\ell \in S_\ell)$.

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Cor. $||e^{\boldsymbol{a}}v_{\ell\varpi_2}||^2 \in 1 + qA$ by the previous theorem.

Lemma

$||e^{\boldsymbol{a}} v_{\ell\varpi_2}||^2$ (in $V(\ell\varpi_2)$) $= ||e^{\boldsymbol{a}}(w_1)^{\otimes \ell}||^2$ (in $\left(W^{2,1}\right)^{\otimes \ell}$).

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pf.) (i) $^{\exists}$inj. hom. $V(\ell\varpi_2) \hookrightarrow V(\varpi_2)^{\otimes\ell}$ [Nakajima].

(ii) $V(\varpi_2) \cong \mathbb{Q}[z, z^{-1}] \otimes W^{2,1} \Rightarrow$ $^{\exists}$hom. $V(\varpi_2) \overset{(z=1)}{\twoheadrightarrow} W^{2,1}$.

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By combining this with the previous corollary, we have $||e^{\boldsymbol{a}}(w_1)^{\otimes\ell}||^2 \in 1 + qA$, and hence it suffices to show the following:

Lemma

$||e^{\boldsymbol{a}}(w_1)^{\otimes\ell}||^2$ (in $(W^{2,1})^{\otimes\ell}$) $= ||e^{\boldsymbol{a}} w_\ell||^2$ (in $W^{2,\ell}$).

For simplicity, assume $\ell = 2$.

pf of $||e^{\boldsymbol{a}}(w_1)^{\otimes 2}||^2$ (in $(W^{2,1})^{\otimes 2}$) $= ||e^{\boldsymbol{a}} w_2||^2$ (in $W^{2,2}$)

$$||e^{\boldsymbol{a}}(w_1)^{\otimes 2}||^2 = ||\sum_{\boldsymbol{b}} q^{c(\boldsymbol{b})} e^{\boldsymbol{a}-\boldsymbol{b}} w_1 \otimes e^{\boldsymbol{b}} w_1||^2$$

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$$= \sum_{\boldsymbol{b},\boldsymbol{b}'} q^{c(\boldsymbol{b})+c(\boldsymbol{b}')} (e^{\boldsymbol{a}-\boldsymbol{b}} w_1, e^{\boldsymbol{a}-\boldsymbol{b}'} w_1)(e^{\boldsymbol{b}} w_1, e^{\boldsymbol{b}'} w_1)$$

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recall $\big(R(u), R(v)\big) = \big(u, R(v)\big)'$, where $R\colon W_q^2 \otimes W_{q^{-1}}^2 \xrightarrow{R} W_{q^{-1}}^2 \otimes W_q^2$.

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$$= \Big(\textstyle\sum_{\boldsymbol{b}} q^{c(\boldsymbol{b})+d(\boldsymbol{b})} e^{\boldsymbol{a}-\boldsymbol{b}} w_1 \otimes e^{\boldsymbol{b}} w_1, \sum_{\boldsymbol{b'}} q^{c(\boldsymbol{b'})-d(\boldsymbol{b'})} e^{\boldsymbol{a}-\boldsymbol{b'}} w_1 \otimes e^{\boldsymbol{b'}} w_1\Big)'$$

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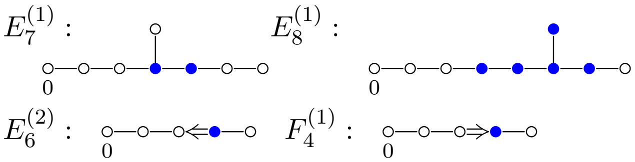
$= \sum_{\boldsymbol{b}} q^{2c(\boldsymbol{b})} ||e^{\boldsymbol{a}-\boldsymbol{b}} w_1||^2 ||e^{\boldsymbol{b}} w_1||^2$. □

pf of $||e_i u||^2 \in q^{-2\langle h_i, \mathrm{wt}(u)\rangle - 1} A$ is in a similar spirit.

Future work: remaining cases



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$\rightsquigarrow$ no explicit, closed formula for dec. $W^{r,\ell} \cong \bigoplus V_0\big(\mathrm{wt}(u)\big)$ so far.

Hence it is difficult to find the vectors $\{u_k\}$ in the criterion.

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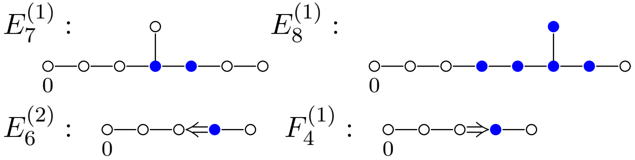

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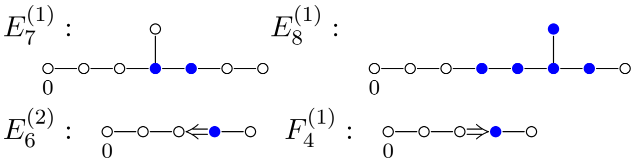

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<u>Q.</u> Can we find an algorithm:

(vectors of $W^{r,\ell}$ in the criterion) $\to$ (vectors of $W^{r,\ell+1}$ in the criterion)

corresponding to the Kleber algorithm?