

Equivalence between module categories over quiver Hecke algebras and Hernandez-Leclerc's subcategories

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Quantum Groups and Cluster Algebras,

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Summary of today's result

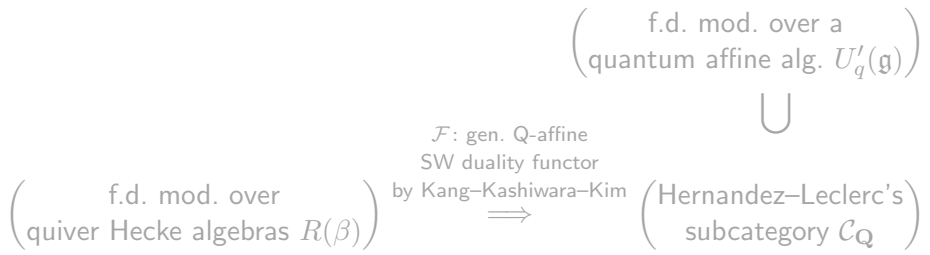
$$\begin{array}{ccc}
 & & \left(\begin{array}{c} \text{f.d. mod. over a} \\ \text{quantum affine alg. } U'_q(\mathfrak{g}) \end{array} \right) \\
 & & \cup \\
 & \mathcal{F}: \text{ gen. Q-affine} \\
 & \text{SW duality functor} \\
 \left(\begin{array}{c} \text{f.d. mod. over} \\ \text{quiver Hecke algebras } R(\beta) \end{array} \right) & \xrightarrow{\text{by Kang-Kashiwara-Kim}} & \left(\begin{array}{c} \text{Hernandez-Leclerc's} \\ \text{subcategory } \mathcal{C}_{\mathbf{Q}} \end{array} \right)
 \end{array}$$

Theorem ([N])

In a general affine type, \mathcal{F} gives an equivalence of two monoidal categories.

In untwisted *ADE* types, this was previously proved by Fujita.

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- Notation**
- $A\text{-mod}$: cat. of f.g. A -modules
 - $A\text{-fmod}$: cat. of f.d. A -modules

Quiver Hecke algebras $R(\beta)$

Khovanov–Lauda [KL09] and Rouquier [Rou08] defined independently.

Given a Kac-Moody \mathfrak{g} whose Cartan matrix is A

$\rightsquigarrow R(\beta)$: **quiver Hecke algebras** (family of algebras, $\beta \in Q^+ = \sum_i \mathbb{Z}_{\geq 0} \alpha_i$)

○ $R(\beta)$ are \mathbb{Z} -graded algebras,

○ $M \in R(\beta)\text{-fmod}^{\text{gr}}$, $M' \in R(\beta')\text{-fmod}^{\text{gr}}$,

$\rightsquigarrow M \circ M' \in R(\beta + \beta')\text{-fmod}^{\text{gr}}$: **convolution product**

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Theorem ([KL09],[Rou08])

$$\bigoplus_{\beta} K(R(\beta)\text{-fmod}^{\text{gr}}) \cong U_{\mathbb{Z}}^-(\mathfrak{g})^{\vee}: \text{int. form of the dual of the half of } U_q(\mathfrak{g})$$

(as $\mathbb{Z}[q^{\pm 1}]$ -algebra)

Theorem ([Varagnolo-Vasserot, 11], [Rouquier, 12])

\mathfrak{g} : symmetric \Rightarrow the isom. sends simples to the upper global basis.

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\cup \cup
 {simples} \rightarrow {upper global basis}

By specializing at $q = 1$, we obtain the following.

Corollary

If \mathfrak{g} is a simple Lie algebra of type ADE ,

(i) $\bigoplus_{\beta} \mathbb{C} \otimes_{\mathbb{Z}} K(R(\beta)\text{-fmod}^0) \cong \mathbb{C}[N]$,

where $R(\beta)\text{-fmod}^0$: cat. of f.d. mod. on which x_k 's act nilpotently
 (obtained from graded ones by forgetting the gradings)

$\mathbb{C}[N]$: coordinate ring of the unipotent group associated with \mathfrak{g} .

(ii) This isom. sends simples to (the specialization of) upper global basis.

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There is another algebra categorifying the same things!

Hernandez–Leclerc's subcategory

\mathfrak{g} : simple Lie algebra of type ADE ,

R^+ : positive roots of \mathfrak{g}

$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t^{\pm 1}] \oplus \mathbb{C}K$: untwisted affine Lie alg. assoc. with \mathfrak{g}

$\rightsquigarrow U'_q(\widehat{\mathfrak{g}})$: quantum group of $\widehat{\mathfrak{g}}$ (**quantum affine algebra**),

$\mathcal{C}_{\widehat{\mathfrak{g}}} := U'_q(\widehat{\mathfrak{g}})\text{-fmod}$: cat. of f.d. $U'_q(\widehat{\mathfrak{g}})$ -module

Given a Dynkin quiver of type \mathfrak{g} (with additional data), they defined a map $Q: R^+ \ni \alpha \mapsto V^\alpha \in \mathcal{C}_{\hat{\mathfrak{g}}}$: simple (fundamental) mod. using the Auslander-Reiten quiver of its path algebra

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Ex. ($\mathfrak{g} = A_3$)

Given a Dynkin quiver of type g (with additional data), they defined a map $Q: R^+ \ni \alpha \mapsto V^\alpha \in \mathcal{C}_{\hat{g}}$: simple (fundamental) mod. using the Auslander-Reiten quiver of its path algebra

Ex. ($g = A_3$)

$\mathcal{C}_{\hat{g}} \supseteq \mathcal{C}_Q$: containing V^α ($\alpha \in R^+$), stable under \otimes , ext., subquot..
(\mathcal{C}_Q in Oya's talk: "fundamental domain" for dual operations)

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Theorem (Hernandez–Leclerc, '15)

$\mathbb{C} \otimes_{\mathbb{Z}} K(\mathcal{C}_Q) \cong \mathbb{C}[N]$ as a \mathbb{C} -algebra, and this sends simples to (the specialization of) upper global basis.

$$\bigoplus_{\beta} \mathbb{C} \otimes_{\mathbb{Z}} K(R(\beta)\text{-fmod}^0) \cong \mathbb{C}[N] \cong \mathbb{C} \otimes_{\mathbb{Z}} K(\mathcal{C}_Q)$$

(simples) \leftrightarrow (gl. basis) \leftrightarrow (simples)

Q. Is there a functor between $R(\beta)\text{-fmod}^0$ and \mathcal{C}_Q inducing this isomorphism?

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Type A [Chari–Pressley, Cherednik, Ginzburg–Varagnolo–Vasserot]

$R(\beta)\text{-fmod}^0 \cong H_q^{\text{aff}}(d)\text{-fmod}$ (affine Hecke algebra)

$\mathbb{V}^{\otimes d}$: $(U'_q(\widehat{\mathfrak{sl}}_n), H_q^{\text{aff}}(d))\text{-bimodule}$

$\Rightarrow H_q^{\text{aff}}(d)\text{-fmod} \ni M \mapsto \mathbb{V}^{\otimes d} \otimes_{H_q^{\text{aff}}(d)} M \in \mathcal{C}_{\widehat{\mathfrak{sl}}_n}$

(quantum affine Schur–Weyl duality functor)

Kang–Kashiwara–Kim's construction of functors

[KKK18]: construction of functors in general setting

[KKK15]: application of the results in [KKK18] to HL subcategories
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$U'_q(\mathfrak{g})$: quantum affine algebra of a general affine type, $\mathcal{C}_{\mathfrak{g}} := U'_q(\mathfrak{g})\text{-fmod}$

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\rightsquigarrow define a Cartan matrix $A = (a_{ij})_{i,j \in J}$ by

$$a_{ij} = \begin{cases} 2 & (i = j), \\ -b_{ij} - b_{ji} & (i \neq j), \end{cases} \text{ where}$$

$$b_{ij} = (\text{deg. of pole of } V_i \otimes (V_j[z^{\pm 1}])) \xrightarrow{R^{\text{norm}}} (V_j(z)) \otimes V_i \text{ at } z = 1).$$

$\rightsquigarrow \{R(\beta)\}_{\beta \in Q^+}$: quiver Hecke algebras assoc. with A

Then we construct a $(U'_q(\mathfrak{g}), R(\beta))$ -bimodule as follows.

$V_i (i \in J) \rightsquigarrow \widehat{V}_i = V_i[[w]]$: a completed affinization $(U'_q(\mathfrak{g})\text{-module})$

For $\beta \in Q^+$, $\widehat{V}^{\otimes \beta} = \bigoplus_{\alpha_{i_1} + \dots + \alpha_{i_p} = \beta} \widehat{V}_{i_1} \hat{\otimes} \dots \hat{\otimes} \widehat{V}_{i_p}$.

$U'_q(\mathfrak{g}) \curvearrowright \widehat{V}^{\otimes \beta} \curvearrowleft R(\beta)$ defined using R -matrices

$\rightsquigarrow \mathcal{F}_\beta: R(\beta)\text{-fmod}^0 \rightarrow \mathcal{C}_\mathfrak{g}, \quad M \mapsto \widehat{V}^{\otimes \beta} \otimes_{R(\beta)} M$

$\mathcal{F} = \bigoplus_\beta \mathcal{F}_\beta: \bigoplus_\beta R(\beta)\text{-fmod}^0 \rightarrow \mathcal{C}_\mathfrak{g}$: **gene'd Q-aff. SW duality functor**

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Theorem ([KKK18])

(i) \mathcal{F} is monoidal ($\mathcal{F}(M \circ M') \cong \mathcal{F}(M) \otimes \mathcal{F}(M')$, etc.).

(ii) If $\{R(\beta)\}$ are of type ADE , \mathcal{F} is exact.

In [KKK15], the results of [KKK18] were applied in untwisted *ADE* types ($\mathfrak{g} = \widehat{\mathfrak{g}}$), and gave a positive answer to the previous question (i.e., existence of a functor inducing $\bigoplus_{\beta} K(R(\beta)\text{-fmod}^0)_{\mathbb{C}} \xrightarrow{\cong} K(\mathcal{C}_Q)_{\mathbb{C}}$)

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recall In the construction of \mathcal{C}_Q , a map $R^+ \ni \alpha \mapsto V^\alpha \in \mathcal{C}_{\widehat{\mathfrak{g}}}$ is used.

Take $\{V^{\alpha_i}\}_{i \in J}$ as the given data $\rightsquigarrow \mathcal{F}: \bigoplus_{\beta} R(\beta)\text{-fmod}^0 \rightarrow \mathcal{C}_{\widehat{\mathfrak{g}}}$.

○ In this case, $R(\beta)$ is of type $\mathfrak{g} \Rightarrow \mathcal{F}$ is exact.

○ The image of $\mathcal{F}: \bigoplus_{\beta} R(\beta)\text{-fmod}^0 \rightarrow \mathcal{C}_{\widehat{\mathfrak{g}}}$ is contained in \mathcal{C}_Q .

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Theorem ([KKK15])

In this case, the gene'd QASW duality functor $\mathcal{F}: \bigoplus_{\beta} R(\beta)\text{-fmod}^0 \rightarrow \mathcal{C}_Q$,

which is monoidal and exact, gives one-to-one corresp. between simples.

($\Rightarrow \bigoplus_{\beta} K(R(\beta)\text{-fmod}^0) \xrightarrow{\sim} K(\mathcal{C}_Q)$)

$\mathcal{F}: \bigoplus_{\beta} R(\beta)\text{-fmod}^0 \rightarrow \mathcal{C}_Q$ is monoidal, exact, gives one-to-one correspondence between simples.

Natural problems

(i) Is this an equivalence?

(ii) Is there a generalization to the cases other than untwisted ADE types?

Both problems have been solved affirmatively!

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Theorem ([Fujita, 17], [Fujita, 20])

The generalized QASW duality functor $\mathcal{F}: \bigoplus_{\beta} R(\beta)\text{-fmod}^0 \rightarrow \mathcal{C}_Q$ gives an equivalence of monoidal categories (in untwisted ADE types).

In the proof of [Fujita, 17], he used the geometric representation theory on quiver varieties and the theory of affine highest weight categories (we will return to this result later).

generalization to non-ADE cases

Consider the following quantum aff. alg. $U'_q(\mathfrak{g})$ and simple Lie alg. \mathfrak{g} :

$U'_q(\mathfrak{g})$	$B_n^{(1)}$	$C_n^{(1)}$	$F_4^{(1)}$	$G_2^{(1)}$	$A_n^{(2)}$	$D_n^{(2)}$	$E_6^{(2)}$	$D_4^{(3)}$
\mathfrak{g}	A_{2n-1}	D_{n+1}	E_6	D_4	A_n	D_n	E_6	D_4

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$R_{\mathfrak{g}}^+$: the set of positive roots of \mathfrak{g} ,

There is a way to define a map $R_{\mathfrak{g}}^+ \ni \alpha \mapsto V^\alpha \in \mathcal{C}_{\mathfrak{g}}$,

$\rightsquigarrow \mathcal{C}_Q = \langle V^\alpha \rangle_{\otimes, \text{ext.}, \text{subquot.}}$ (**Hernandez–Leclerc’s subcategory**)

\rightsquigarrow functor $\mathcal{F}: \bigoplus_{\beta} R(\beta)\text{-fmod}^0 \rightarrow \mathcal{C}_Q$ ($R(\beta)$: quiver Hecke of type \mathfrak{g})

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Theorem ([KKK16], [Kashiwara–Oh, 19], [Oh–Scrimshaw, 19])

In all the above cases, the gene’d QASW duality functor \mathcal{F} is monoidal, exact, and gives one-to-one correspondence between simple modules.

$$\left(\Rightarrow \bigoplus_{\beta} K(R(\beta)\text{-fmod}^0) \xrightarrow{\sim} K(\mathcal{C}_Q). \right)$$

Summary

$U'_q(\mathfrak{g})$	monoidal	exact	bij. of simples	equiv.
ADE	○	○	○	○
others	○	○	○	?

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$U'_q(\mathfrak{g})$	monoidal	exact	bij. of simples	equiv.
ADE	○	○	○	○
others	○	○	○	?

Theorem ([N])

In general types, the gene'd QASW duality functor \mathcal{F} gives an equivalence of monoidal categories $\bigoplus_{\beta} R(\beta)\text{-fmod}^0$ and \mathcal{C}_Q .

Proof to [Conjecture 5.7, KKK16], [Conjecture 6.11, KO19].

Corollary

Let $\mathfrak{g}^{(1)}$: untwisted ADE , $\mathfrak{g}^{(t)}$: twisted, ${}^L\mathfrak{g}^{(t)}$: the Langland dual of $\mathfrak{g}^{(t)}$
 (e.g. $\mathfrak{g}^{(1)} = A_{2n-1}^{(1)}$, $\mathfrak{g}^{(2)} = A_{2n-1}^{(2)}$, ${}^L\mathfrak{g}^{(2)} = B_n^{(1)}$)

$\mathcal{C}_{Q^{(1)}}$, $\mathcal{C}_{Q^{(t)}}$, \mathcal{C}_{LQ} : corresponding HL subcategories

Corollary

The monoidal categories $\mathcal{C}_{Q^{(1)}}$, $\mathcal{C}_{Q^{(t)}}$, \mathcal{C}_{LQ} are mutually equivalent.

\therefore The corresponding quiver Hecke algebras $R(\beta)$ are the same. □

Ex.

$$\begin{array}{c}
 \mathcal{C}_{Q^{(1)}} \subseteq \mathcal{C}_{A_{2n-1}^{(1)}} \\
 \uparrow \wr \\
 \mathcal{C}_{A_{2n-1}^{(2)}} \supseteq \mathcal{C}_{Q^{(2)}} \xleftarrow{\sim} \bigoplus_{\beta} R^{A_{2n-1}}(\beta)\text{-fmod}^0 \xrightarrow{\sim} \mathcal{C}_{LQ} \subseteq \mathcal{C}_{B_n^{(1)}}
 \end{array}$$

strategy of the proof of $\mathcal{F}: \bigoplus R(\beta)\text{-fmod}^0 \xrightarrow{\sim} \mathcal{C}_Q$

Fact \mathcal{C}_Q has a block dec. $\mathcal{C}_Q = \bigoplus_{\beta} \mathcal{C}_{Q,\beta}$ ($\beta \in Q^+$) such that

$$\mathcal{F}_{\beta}: R(\beta)\text{-fmod}^0 \rightarrow \mathcal{C}_{Q,\beta}$$

\therefore Enough to prove $\mathcal{F}_{\beta}: R(\beta)\text{-fmod}^0 \xrightarrow{\sim} \mathcal{C}_{Q,\beta}$ for each β .

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From the homological viewpoint, $R(\beta)\text{-fmod}^0$ and $\mathcal{C}_{Q,\beta}$ are too small (e.g., not enough proj.)

$R(\beta) = \bigoplus_{n \in \mathbb{Z}} R(\beta)_n \rightsquigarrow \widehat{R}(\beta) = \prod_n R(\beta)_n$: completion (cf. $\mathbb{C}[z] \rightsquigarrow \mathbb{C}[[z]]$), and consider $\widehat{R}(\beta)\text{-mod}$ instead.

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advantage $\circ \widehat{R}(\beta)\text{-fmod} = R(\beta)\text{-fmod}^0$

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- advantage
- $\widehat{R}(\beta)\text{-fmod} = R(\beta)\text{-fmod}^0$
 - $\widehat{R}(\beta)\text{-mod}$ is **affine highest weight category!**

(a generalization of highest weight cat. by Cline–Parshall–Scott.

$$\Delta(\lambda): \text{standard} \rightarrow L(\lambda): \text{simple} \hookrightarrow \overline{\nabla}(\lambda): \text{proper costandard}$$

$$\mathcal{F}_\beta: R(\beta)\text{-fmod}^0 \rightarrow \mathcal{C}_{Q,\beta},$$

$$M \mapsto \widehat{V}^{\otimes \beta} \otimes_{R(\beta)} M$$

$$(\widehat{V}^{\otimes \beta}: (U'_q(\mathfrak{g}), R(\beta))\text{-bimod.})$$

$$\mathcal{F}_\beta: \widehat{R}(\beta)\text{-fmod} \rightarrow \mathcal{C}_{Q,\beta},$$
$$\cap$$
$$\widehat{R}(\beta)\text{-mod}$$

(aff. h.w.)

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$$\begin{array}{l} \text{extend} \downarrow \\ \mathcal{F}_\beta: \widehat{R}(\beta)\text{-mod} \rightarrow \{U'_q(\mathfrak{g})\text{-modules}\} \quad (\widehat{V}^{\otimes \beta}: (U'_q(\mathfrak{g}), \widehat{R}(\beta))\text{-bimod.}) \\ \quad \quad \quad \cap \\ \quad \quad \quad \text{(aff. h.w.)} \end{array}$$

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$$M \mapsto \widehat{V}^{\otimes \beta} \otimes_{\widehat{R}(\beta)} M$$

extend ↓

$$\mathcal{F}_\beta: \widehat{R}(\beta)\text{-mod} \rightarrow \{U'_q(\mathfrak{g})\text{-modules}\} \quad (\widehat{V}^{\otimes \beta}: (U'_q(\mathfrak{g}), \widehat{R}(\beta))\text{-bimod.})$$

(aff. h.w.)

Theorem ([Fujita, 18])

$A_i\text{-mod}$: affine h.w. ($i = 1, 2$), $F: A_1\text{-mod} \rightarrow A_2\text{-mod}$: exact.

Assume (i) A_i is finitely generated over its center ($i = 1, 2$),

(ii) \exists bijection $f: \Pi_1 \rightarrow \Pi_2$ such that $F(\Delta(\pi)) = \Delta(f(\pi))$,

$F(\overline{\nabla}(\pi)) = \overline{\nabla}(f(\pi))$ for $\forall \pi$.

Then F is an equivalence.

$$\begin{array}{l}
 \mathcal{F}_\beta: \widehat{R}(\beta)\text{-fmod} \rightarrow \mathcal{C}_{Q,\beta}, \\
 \text{extend} \downarrow \quad \cap \\
 \mathcal{F}_\beta: \widehat{R}(\beta)\text{-mod} \rightarrow \text{“}\widehat{\mathcal{C}}_{Q,\beta}\text{”} \\
 \text{(aff. h.w.)} \quad \quad \quad \text{(aff. h.w.?)}
 \end{array}
 \qquad
 \begin{array}{l}
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Then F is an equivalence.

To prove $\mathcal{F}_\beta: R(\beta)\text{-fmod}^0 \xrightarrow{\sim} \mathcal{C}_{Q,\beta}$, enough to show the following:

- (i) Find an algebra A with an algebra homomorphism $\Phi: U'_q(\mathfrak{g}) \rightarrow A$.
- (ii) Show that $\Phi^*|_{A\text{-fmod}}: A\text{-fmod} \rightarrow U'_q(\mathfrak{g})\text{-fmod}$ gives an equivalence between $A\text{-fmod}$ and $\mathcal{C}_{Q,\beta}$.
- (iii) Define $\mathcal{F}'_\beta: \widehat{R}(\beta)\text{-mod} \rightarrow A\text{-mod}$ s.t. $\Phi^* \circ \mathcal{F}'_\beta|_{\widehat{R}(\beta)\text{-fmod}} = \mathcal{F}_\beta$.
- (iv) Show that $A\text{-mod}$ is aff. h.w., and \mathcal{F}'_β gives an equivalence $\mathcal{F}'_\beta: \widehat{R}(\beta)\text{-mod} \xrightarrow{\sim} A\text{-mod}$.

$$\mathcal{F}_\beta: \widehat{R}(\beta)\text{-fmod} \rightarrow A\text{-fmod} \xrightarrow{\Phi^*} \mathcal{C}_{Q,\beta}$$

$$\mathcal{F}'_\beta: \widehat{R}(\beta)\text{-mod} \rightarrow A\text{-mod}$$

(aff. h.w.)

proof in untwisted ADE in [Fujita, 17]

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In [Fujita, 17], he proved these statements with $A = \widehat{\mathcal{K}}^{\mathbb{G}}(Z^\bullet)$ (completed equiv. K -gps of the Steinberg type graded quiver var.)

- (i) $\exists \Phi: U'_q(\mathfrak{g}) \rightarrow \widehat{\mathcal{K}}^{\mathbb{G}}(Z^\bullet)$ by Nakajima,
- (iii) define $\widehat{\mathcal{K}}^{\mathbb{G}}(Z^\bullet) \curvearrowright \widehat{V}^{\otimes \beta}$ geometrically,
- (ii), (iv) work hard (omit)

proof in general types

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There is no quiver var., and we adopt a completely different algebra A .

recall $\widehat{V}^{\otimes\beta}: (U'_q(\mathfrak{g}), \widehat{R}(\beta))\text{-bimod.}$, $\mathcal{F}_\beta(M) := \widehat{V}^{\otimes\beta} \otimes_{\widehat{R}(\beta)} M$

Set $\mathbb{E}^\beta = \text{End}_{\widehat{R}(\beta)^{\text{opp}}}(\widehat{V}^{\otimes\beta})$ (**analog of Schur algebra**).

This \mathbb{E}^β is our A . (i), (iii) are obvious.

Theorem ([N])

Set $\mathbb{E}^\beta = \text{End}_{\widehat{R}(\beta)^{\text{opp}}}(\widehat{V}^{\otimes \beta})$.

- (i) The alg. hom. $\Phi: U'_q(\mathfrak{g}) \rightarrow \mathbb{E}^\beta$ induces an equiv. $\Phi^*: \mathbb{E}^\beta\text{-fmod} \xrightarrow{\sim} \mathcal{C}_{Q,\beta}$.
- (ii) $\mathbb{E}^\beta\text{-mod}$ is aff. h.w., and \mathcal{F}'_β gives an equiv. $\widehat{R}(\beta)\text{-mod} \xrightarrow{\sim} \mathbb{E}^\beta\text{-mod}$.

In the proof, the **affine cellular str.** of (a quotient of) $U'_q(\mathfrak{g})$ and \mathbb{E}^β are used.

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Thank you for your attention!