Generalized quantum affine Schur-Weyl duality & categorical equivalence

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0. Introduction

• Schur-Weyl duality (Schur, early 20th century)

$\mathfrak{sl}_n := \mathfrak{sl}_n(\mathbb{C}) = \{X \in \mathrm{Mat}_n(\mathbb{C}) \mid \mathrm{tr}\, X = 0\}$ $([X,Y]=XY-YX)$

$\mathfrak{sl}_n \curvearrowright V := \mathbb{C}^n \rightsquigarrow \mathfrak{sl}_n \curvearrowright V^{\otimes d}$ $(d \in \mathbb{Z}_{>0})$ $\left(X(v_1 \otimes \cdots \otimes v_d) = \sum_{i=1}^{d} v_1 \otimes \cdots \otimes X v_i \otimes \cdots \otimes v_d\right)$

commute

$\rightsquigarrow \mathfrak{sl}_n \curvearrowright V^{\otimes d} \curvearrowleft \mathfrak{S}_d$ (symmetric group)

In other words, $V^{\otimes d}$ is an $(\mathfrak{sl}_n, \mathfrak{S}_d)$-bimod.

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In many cases, it is more useful to consider bimod. over associative alg!

$\mathfrak{S}_d \rightsquigarrow \mathbb{C}[\mathfrak{S}_d]$ (group alg.) $(M : \mathfrak{S}_d\text{-mod} \Leftrightarrow M : \mathbb{C}[\mathfrak{S}_d]\text{-mod})$

$\mathfrak{sl}_n \rightsquigarrow U(\mathfrak{sl}_n)$ (universal enveloping alg.)

$:= T(\mathfrak{sl}_n) / \langle X \otimes Y - Y \otimes X - [X,Y] \mid X, Y \in \mathfrak{sl}_n \rangle$ $(M : \mathfrak{sl}_n\text{-mod} \Leftrightarrow M : U(\mathfrak{sl}_n)\text{-mod})$

tensor alg. $T(\mathfrak{sl}_n) = \bigoplus_i \mathfrak{sl}_n^{\otimes i}$

Then $V^{\otimes d}$ is an $(U(\mathfrak{sl}_n), \mathbb{C}[\mathfrak{S}_d])$-bimod.

$\rightsquigarrow F : \mathbb{C}[\mathfrak{S}_d]\text{-fmod} \ni M \mapsto F(M) := V^{\otimes d} \otimes_{\mathbb{C}[\mathfrak{S}_d]} M \in U(\mathfrak{sl}_n)\text{-fmod}$

(cat. of f.d. mod.)

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<u>Thm</u> Assume $n \geq d$.

1. For d_1, d_2 s.t. $d = d_1 + d_2$ and $M \in \mathbb{C}[\mathfrak{S}_{d_1}]$-fmod, $N \in \mathbb{C}[\mathfrak{S}_{d_2}]$-fmod, if we set
$$M \circ N := \mathbb{C}[\mathfrak{S}_d] \otimes_{\mathbb{C}[\mathfrak{S}_{d_1}] \otimes \mathbb{C}[\mathfrak{S}_{d_2}]} (M \otimes N),$$
then $F(M \circ N) \simeq F(M) \otimes F(N)$.

2. Set $E := \mathrm{End}_{\mathbb{C}[\mathfrak{S}_d]}(V^{\otimes d})$. Then the canonical alg. hom $U(\mathfrak{sl}_n) \to E$ is **Surjective**.

3. F gives an equiv.
$$\mathbb{C}[\mathfrak{S}_d]\text{-fmod} \xrightarrow{\sim} E\text{-fmod} \underset{\text{full sub}}{\subseteq} U(\mathfrak{sl}_n)\text{-fmod}$$

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- quantum SW duality (Jimbo, '86)

$$\begin{array}{ccc} U_q(\mathfrak{sl}_n) : \text{quantum gp} & \leftrightarrow & \mathcal{H}_d(q) : \text{Iwahori-Hecke alg.} \\ \downarrow q \to 1 & & \downarrow q \to 1 \\ U(\mathfrak{sl}_n) & & \mathbb{C}[\mathfrak{S}_d] \end{array}$$

- quantum affine SW duality

(Chari-Pressley, Cherednik, Ginzburg-Varagnolo-Vasserot, around '95)

$$\begin{array}{ccc} U_q(\widehat{\mathfrak{sl}}_n) : \text{quantum affine alg.} & & \\ \downarrow q \to 1 & & \\ U(\widehat{\mathfrak{sl}}_n) & \leftrightarrow & \mathcal{H}_d^{\mathrm{aff}}(q) : \text{affine Hecke alg.} \\ & & \downarrow q \to 1 \\ (\widehat{\mathfrak{sl}}_n = \mathfrak{sl}_n \otimes \mathbb{C}[t^{\pm 1}]) & & \mathbb{C}[\mathfrak{S}_d] \ltimes \mathbb{C}[t_1^{\pm 1}, \dots, t_d^{\pm 1}] \end{array}$$

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$U(\widehat{\mathfrak{sl}_n})$ $\longleftrightarrow$ $\mathcal{H}_d^{\mathrm{aff}}(q)$: affine Hecke alg.

$\downarrow q\to 1$

$(\widehat{\mathfrak{sl}_n}=\mathfrak{sl}_n\otimes\mathbb{C}[t^{\pm 1}])$ $\quad$ $\mathbb{C}[\mathfrak{S}_d]\ltimes\mathbb{C}[t_1^{\pm 1},\cdots,t_d^{\pm 1}]$

- Gen'd quantum affine SW duality (Kang-Kashiwara-Kim, '18)

$\mathfrak{g}$: simple Lie alg. (e.g. $\mathfrak{sl}_n, \mathfrak{so}_n, \mathfrak{sp}_{2n}, \cdots$)

$U_q(\hat{\mathfrak{g}})$: quantum affine alg. of general type

$\downarrow q\to 1$ $\quad\quad$ $\updownarrow$

$U(\hat{\mathfrak{g}})$

$R(\beta)$: quiver Hecke alg.

$(U_q(\hat{\mathfrak{g}}), R(\beta))$-bimod.

$\rightsquigarrow$ functor from "$R(\beta)$-fmod" to $U_q(\hat{\mathfrak{g}})$-fmod

Main Thm

In certain special cases, this functor gives an equiv. between suitable full subcategories. $\lrcorner$

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Plan

§1 quantum affine alg.

§2 quiver Hecke alg.

§3 Kang-Kashiwara-Kim's construction of functor

§4 Main Thm

§5 proof (if time permits)

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type A_n B_n C_n D_n

fin. index set

$\mathfrak{g} = \langle e_i, f_i, h_i \mid i \in I \rangle$ Chevalley generators

$\hat{\mathfrak{g}} := \mathfrak{g} \otimes \mathbb{C}[t^{\pm 1}]$ $([X \otimes f, Y \otimes g] = [X, Y] \otimes fg)$

(loop)

$U_q(\hat{\mathfrak{g}})$: quantum affine alg.

$\downarrow q \to 1$ $(q \in \mathbb{C}^\times$: not a root of 1)

$U(\hat{\mathfrak{g}})$

	gen	def. rel.
$U(\hat{\mathfrak{g}})$	$e_i \otimes t^k, f_i \otimes t^k, h_i \otimes t^k$ $(i \in I, k \in \mathbb{Z})$	$[h_i \otimes t^k, e_i \otimes t^m] = 2 e_i \otimes t^{k+m}$ etc.
$U_q(\hat{\mathfrak{g}})$	$e_{i,k}, f_{i,k}, h_{i,k}$	$[h_{i,k}, e_{i,m}] = \frac{q^{2m} - q^{-2m}}{m(q - q^{-1})} e_{i,k+m}$, etc.

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Properties

$U_q(\hat{\mathfrak{g}})$ is a Hopf alg.

i.e. $\exists \Delta : U_q(\hat{\mathfrak{g}}) \to U_q(\hat{\mathfrak{g}}) \otimes U_q(\hat{\mathfrak{g}})$ coproduct

$\exists S : U_q(\hat{\mathfrak{g}}) \to U_q(\hat{\mathfrak{g}})^{op}$ antipode

satisfying certain conditions.

$\rightsquigarrow U_q(\hat{\mathfrak{g}})$-fmod : **rigid monoidal cat.**

i.e. $M, N \in U_q(\hat{\mathfrak{g}})$-fmod

$\rightsquigarrow$
- $M \otimes N \in U_q(\hat{\mathfrak{g}})$-fmod
- $\exists M^*$: right dual, ${}^*M$: left dual, etc. ┘

<u>Rem</u> $U_q(\hat{\mathfrak{g}})$-fmod is **not semisimple** ┘

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§2 quiver Hecke alg.

$\mathfrak{g}$: simple Lie alg (or Kac-Moody Lie alg.) + additional data
w/ index set I

$\rightsquigarrow \{R(\beta)\}_{\beta \in \mathbb{Z}_{\geq 0}^I}$ quiver Hecke alg
(or Khovanov-Lauda-Rouquier alg.)

(a family of alg. parametrized by $\mathbb{Z}_{\geq 0}^I$)

For $\beta = (\beta_i)_{i \in I} \in \mathbb{Z}_{\geq 0}^I$ w/ $\sum_i \beta_i = m$,

set $I^\beta := \{ \underset{\in I^m}{\mathbf{i} = (i_1, \cdots, i_m)} \mid \# \{ i_k = i \} = \beta_i \ (\forall i) \}$

$$\mathbb{C}[\underset{\substack{\cup\\ s_\ell}}{\mathfrak{S}_m}] \ltimes \Big(\underbrace{\bigoplus_{\mathbf{i} \in I^\beta}}_{\text{deform}} \mathbb{C}[x_1, \cdots, x_m] \underbrace{e(\mathbf{i})}_{\text{idempotent}} \Big)$$

$$R(\beta) = \langle e(\mathbf{i}), x_k, \tau_\ell \mid \mathbf{i} \in I^\beta, 1 \leq k \leq m, 1 \leq \ell \leq m-1 \rangle$$

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Ex. $\mathfrak{g} = \mathfrak{sl}_{n+1}$ $(I=\{1,2,\cdots,n\})$

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$$x_k x_\ell = x_\ell x_k,\quad \tau_\ell^2 e(\mathbf{i}) = \begin{cases} 0 & (i_\ell = i_{\ell+1}) \\ \pm(x_\ell - x_{\ell+1})e(\mathbf{i}) & (|i_\ell - i_{\ell+1}| = 1) \\ e(\mathbf{i}) & (\text{o.w.}) \end{cases}$$

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∎

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Properties

- $R(\beta)$: $\mathbb{Z}$-graded alg
- $\beta_1, \beta_2 \in \mathbb{Z}^I_{\ge 0}$, $M \in R(\beta_1)$-fgmod, $N \in R(\beta_2)$-fgmod

$\rightsquigarrow M \circ N := R(\beta_1 + \beta_2) \otimes_{R(\beta_1) \otimes R(\beta_2)} (M \otimes N)$ convolution product

$\therefore \bigoplus_{\beta \in \mathbb{Z}^I_{\ge 0}} R(\beta)$-fgmod : **monoidal cat.**

$\xrightarrow{\text{Grothendieck group}}$ $K\left(\bigoplus_{\beta \in \mathbb{Z}^I_{\ge 0}} R(\beta)\text{-fgmod}\right)$ has a $\mathbb{Z}[q^{\pm 1}]$-alg. structure via

- $[M] \cdot [N] = [M \circ N]$
- $q[M] = [M\underbrace{[1]}_{\text{grading shift}}]$

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$\rightsquigarrow$ Grothendieck ring: $K(\bigoplus_{\beta \in \mathbb{Z}_{\geq 0}^I} R(\beta)\text{-fgmod})$ has a $\mathbb{Z}[q^{\pm 1}]$-alg.

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Thm

1. (Khovanov-Lauda, Rouquier)

$K(\bigoplus_{\beta} R(\beta)\text{-fgmod}) \simeq U_{\mathbb{Z}}^{-}(\mathfrak{g})^{\vee}$ as $\mathbb{Z}[q^{\pm 1}]$-alg.

$U_{\mathbb{Z}}^{-}(\mathfrak{g})^{\vee} \subset U_q(\mathfrak{g})$: $\mathbb{Z}[q^{\pm 1}]$-subalg.

2. (Varagnolo-Vasserot, Rouquier)

If $\mathfrak{g}$ is simply-laced (e.g. $\mathfrak{sl}_n$, $\mathfrak{so}_{2n}$), the above isom sends the classes of self-dual simples to **upper global base**.

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$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad U_q(\mathfrak{g})$

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If $\mathfrak{g}$ is simply-laced (e.g. $\mathfrak{sl}_n$, $\mathfrak{so}_{2n}$), the above isom sends the classes of self-dual simples to **upper global base**.

§3. Kang-Kashiwara-Kim's construction of functors

$\mathfrak{g}$: simple Lie alg. $\rightsquigarrow$ $U_q(\hat{\mathfrak{g}})$: quantum affine alg.

Let J be a set, and $\{V_j\}_{j\in J}$ a family of real simples in $U_q(\hat{\mathfrak{g}})$-fmod.

(f.d. simple $U_q(\hat{\mathfrak{g}})$-mod is real $\overset{\text{def}}{\Leftrightarrow}$ $V\otimes V$: simple)

step 1 From $\{V_j\}_{j\in J}$, determine a quiver Hecke alg. $\{R(\beta)\}$

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Let $i, j \in J$.

<u>Fact</u> $V_i\otimes V_j \simeq V_j\otimes V_i$ is not necessarily true,

but $\exists R : V_i\otimes V_j(z) \xrightarrow{\sim} V_j(z)\otimes V_i$

normalized R-matrix

Let $d_{ij}\in\mathbb{Z}_{\geq 0}$ be the order of the pole of R at $z=1$

<u>Rem</u> $d_{ij}=0 \Leftrightarrow V_i\otimes V_j \xrightarrow{\sim} V_j\otimes V_i$

Define a graph Ω w/

- vertices J
- d_{ij} edges between i & j $\quad \underset{i}{\circ} \overset{d_{ij}}{\equiv} \underset{j}{\circ}$

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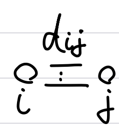
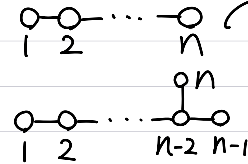


Let $\mathfrak{g}_1$ be the Lie alg. whose Dynkin diagram is Ω

e.g.) Ω $\quad$ $\mathfrak{g}_1$



$\rightsquigarrow \mathfrak{sl}_{n+1}$

$\rightsquigarrow \mathfrak{so}_{2n}$

$\rightsquigarrow \{R(\beta)\}_{\beta \in \mathbb{Z}_{\geq 0}^J}$: quiver Hecke alg. assoc. w/ $\mathfrak{g}_1$

Step 2 For each $\beta \in \mathbb{Z}_{\geq 0}^J$, construct a $(U_q(\hat{\mathfrak{g}}), R(\beta))$-bimod. $\widehat{V}^{\otimes \beta}$

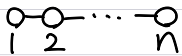
For $j \in J$, set $\widehat{V}_j := V_j[z^{\pm 1}] \otimes_{\mathbb{C}[z^{\pm 1}]} \mathbb{C}[[w]]$ $(w = z - 1)$

$U_q(\hat{\mathfrak{g}}) \curvearrowright \widehat{V}^{\otimes \beta} := \bigoplus_{\underline{i} \in J^\beta} \widehat{V}_{i_1} \hat{\otimes} \cdots \hat{\otimes} \widehat{V}_{i_m} \curvearrowleft R(\beta)$

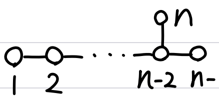
via normalized R-matrix

Let $\mathfrak{g}_1$ be the Lie alg. whose Dynkin diagram is Ω

e.g.) Ω $\rightsquigarrow$ $\mathfrak{g}_1$



$\rightsquigarrow$ $\mathfrak{sl}_{n+1}$



$\rightsquigarrow$ $\mathfrak{so}_{2n}$

$\rightsquigarrow \{R(\beta)\}_{\beta \in \mathbb{Z}_{\geq 0}^J}$: quiver Hecke alg. assoc. w/ $\mathfrak{g}_1$

Step 2) For each $\beta \in \mathbb{Z}_{\geq 0}^J$, construct a $(U_q(\hat{\mathfrak{g}}), R(\beta))$-bimod. $\widehat{V}^{\otimes\beta}$

For $j \in J$, set $\widehat{V}_j := V_j[z^{\pm 1}] \otimes_{\mathbb{C}[z^{\pm 1}]} \mathbb{C}[[w]]$ $(w = z-1)$

$U_q(\hat{\mathfrak{g}}) \curvearrowright \widehat{V}^{\otimes\beta} := \bigoplus_{i \in J^\beta} \widehat{V}_{i_1} \hat{\otimes} \cdots \hat{\otimes} \widehat{V}_{i_m} \curvearrowleft R(\beta)$

via normalized R-matrix

$\therefore \widehat{V}^{\otimes\beta} : (U_q(\hat{\mathfrak{g}}), R(\beta))$-bimod.

$R(\beta)\text{-fmod}_0 := \{M \in R(\beta)\text{-fmod} \mid x_i \text{ acts nilpotently on } M \ (\forall i)\}$

$\underset{\text{full sub}}{\subseteq} R(\beta)\text{-fmod}$

$\mathcal{F}_\beta : R(\beta)\text{-fmod}_0 \ni M \mapsto \widehat{V}^{\otimes\beta} \otimes_{R(\beta)} M \in U_q(\hat{\mathfrak{g}})\text{-fmod}.$

$\rightsquigarrow \mathcal{F} = \bigoplus_{\beta \in \mathbb{Z}_{\geq 0}^J} \mathcal{F}_\beta : \bigoplus_\beta R(\beta)\text{-fmod}_0 \to U_q(\hat{\mathfrak{g}})\text{-fmod}$

Generalized quantum affine SW duality functor

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Generalized quantum affine SW duality functor

Thm (KKK)

1. $\mathcal{F}$ is a monoidal functor

(In particular, $\mathcal{F}(M \circ N) \xrightarrow{\sim} \mathcal{F}(M) \otimes \mathcal{F}(N)$)

2. If $\{R(\beta)\}$ is of finite type (i.e. $\mathfrak{g}_1$ $\rightsquigarrow R(\beta)$ is a simple Lie alg.), $\mathcal{F}$ is an exact functor. ⌟

Rem (Histrical background)

affine Hecke alg. $\mathcal{H}_d^{\text{aff}}(q)$ categorifies $U(\mathfrak{sl}_n)$ (LLT-Ariki theory)

$\downarrow$ generalize

quiver Hecke alg. $R(\beta)$ ——⫽—— $U_q(\mathfrak{g})$

On the other hand, $U_q(\widehat{\mathfrak{sl}_n}) \overset{\text{quantum affine SW}}{\longleftrightarrow} \mathcal{H}_d^{\text{aff}}(q)$

generalized? $\rightsquigarrow$ $U_q(\hat{\mathfrak{g}}) \longleftrightarrow R(\beta)$ ⌟

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$\xrightarrow{\text{generalized?}}$ $U_q(\hat{\mathfrak{g}}) \longleftrightarrow R(\beta)$ ⌟

§4 Main Thm

☆ Our main thm will say that the functor $\mathcal{F}$ gives an equiv. $\mathcal{F}: \bigoplus_\beta R(\beta)\text{-fmod}_0 \xrightarrow{\sim} \mathcal{C}_0 \subseteq U_q(\hat{\mathfrak{g}})\text{-fmod}$. for a certain fullsubcat. $\mathcal{C}_0$. ⌟

$V(i,a)$ $(i \in I, a \in \mathbb{C}^\times)$: fundamental modules

(a family of simples in $U_q(\hat{\mathfrak{g}})$-fmod)

Fact (Chari-Pressley)

$U_q(\hat{\mathfrak{g}})$-fmod $\leftarrow$ very huge

$= \langle V(i,a) \mid i \in I, a \in \mathbb{C}^\times \rangle_{\otimes, \text{subquot}, \text{ext}}$ ⌟

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Def (Hernandez-Leclerc)

Take $s_i, r_i \in \mathbb{Z}_{>0}$ suitably for each $i \in I$, and set $\hat{I} := \{(i,\ell) \mid \ell \in s_i + r_i\mathbb{Z}\} \subseteq I \times \mathbb{Z}$.

Then define $\mathcal{C}_{\mathbb{Z}} := \langle V(i, q^{\ell}) \mid (i,\ell) \in \hat{I} \rangle_{\otimes,\text{subquot,ext}}$

Hernandez-Leclerc subcat.

$\subseteq U_q(\hat{\mathfrak{g}})$-fmod.

Ex. $\mathfrak{g} = \mathfrak{sl}_{n+1}$ $(I = \{1, \cdots, n\})$

$\rightsquigarrow \hat{I} = \{(i,\ell) \mid \ell \in i + 2\mathbb{Z}\}$

$\mathfrak{sl}_4$)

$I \backslash \mathbb{Z}$	-3	-2	-1	0	1	2	3	4	5	6
1	o		o		o		o		o	
2		o		o		o		o		o
3	o		o		o		o		o	

Def (Hernandez-Leclerc)

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Prop. Any simple $M \in U_q(\hat{\mathfrak{g}})\text{-fmod}$ is expressed as $M \xrightarrow{\sim} \bigotimes_{b \in \mathbb{C}^{\times}/q^{\mathbb{Z}}} \tau_b^{*} M_b$ w/ $M_b \in \mathcal{C}_{\mathbb{Z}}$

$\uparrow$ an auto on $U_q(\hat{\mathfrak{g}})$

☆ Hence $\mathcal{C}_{\mathbb{Z}}$ has most information of $U_q(\hat{\mathfrak{g}})\text{-fmod}$.

$\uparrow$ still big

Fact

For $\forall (i,\ell) \in \hat{I}$, $V(i, q^{\ell})^{*} \simeq V(i', q^{\ell'})$ for some $\underbrace{(i',\ell')}_{\overset{!!}{D(i,\ell)}} \in \hat{I}$

$\rightsquigarrow D, D^{-1} : \hat{I} \to \hat{I}$ $\lrcorner$

Def For a suitable fundamental domain $\hat{I}_0 \subseteq \hat{I}$ under the action of $D^{\mathbb{Z}}$, set

Core subcat. of $\mathcal{C}_{\mathbb{Z}}$

$\mathcal{C}_0 := \langle V(i, q^{\ell}) \mid (i,\ell) \in \hat{I}_0 \rangle_{\otimes, \text{subquot}, \text{ext}}$ $\lrcorner$

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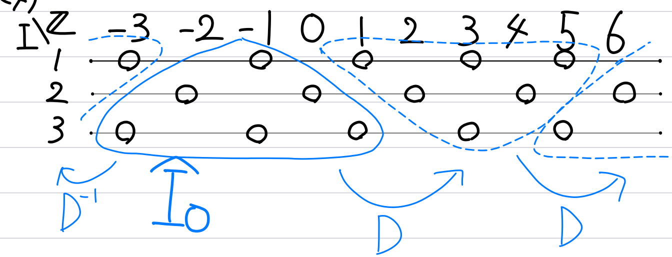
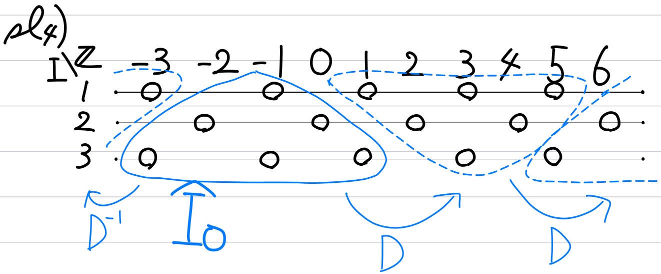


Rem

$\mathcal{C}_{\mathbb{Z}} = \langle V(i,q^a) \mid (i,a)\in \hat{I}_0\rangle_{\otimes,\,\text{suquot},\,\text{ext},\,\text{left/right dual}}$ $\lrcorner$

Ex.



Rem

$\mathscr{C}_{\mathbb{Z}} = \langle V(i, q^{\ell}) \mid (i, a) \in \hat{I}_0 \rangle_{\otimes, \text{ suquot, ext, left/right dual}}$ ⌟

Thm (KKK, Kashiwara-Oh, Oh-Scrimshaw)

1. By applying the KKK construction to the family of simples $\{V(i, q^{\ell}) \mid (i, \ell) \in \underbrace{J}_{\text{suitable subset}} \subseteq \hat{I}_0\}$, a functor $\mathcal{F} : \bigoplus_{\beta} R(\beta)\text{-fmod}_0 \to \mathscr{C}_0 \subseteq U_q(\hat{\mathfrak{g}})\text{-fmod}$ is obtained.

Here the types of $\{R(\beta)\}$ is as follows:

$\mathfrak{g}$	$\mathfrak{sl}_{n+1}$ (A_n)	$\mathfrak{so}_{2n+1}$ (B_n)	$\mathfrak{sp}_{2n}$ (C_n)	$\mathfrak{so}_{2n}$ (D_n)	E_n	F_4	G_2
$\{R(\beta)\}$	〃	$\mathfrak{sl}_{2n+1}$ (A_{2n+1})	$\mathfrak{so}_{2n+2}$ (D_{n+1})	〃	〃	E_6	D_4

Moreover, $\mathcal{F}$ is monoidal & exact.
↑ follows from the previous thm.

2. $\mathcal{F}$ induces a **bij. between simples** of $\bigoplus_{\beta} R(\beta)\text{-fmod}_0$ & $\mathscr{C}_0$ ⌟

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Thm (Fujita: ADE type, N: general type (containing twisted types))

$\mathcal{F}: \bigoplus_{\beta} R(\beta)\text{-fmod}_0 \to \mathcal{C}_0$ is an equivalence.

Cor equivalence of $\mathcal{C}_0$'s over different $U_q(\hat{\mathfrak{g}})$

e.g.) $\mathcal{C}_0^{\mathfrak{sl}_{2n+1}} \simeq \mathcal{C}_0^{\mathfrak{so}_{2n+1}}$, $\mathcal{C}_0^{\mathfrak{sp}_{2n}} \simeq \mathcal{C}_0^{\mathfrak{so}_{2n+2}}$

$\bigoplus_{\beta} R^{\mathfrak{sl}_{2n+1}}(\beta)\text{-fmod}_0$ $\quad$ $\bigoplus_{\beta} R^{\mathfrak{so}_{2n+2}}(\beta)\text{-fmod}_0$

Problem Construct directly the equiv. in Cor.

<u>Thm</u> (Fujita: ADE type, N: general type)

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e.g.) $\mathcal{C}_0^{A^{(1)}_{2n+1}} \simeq \mathcal{C}_0^{A D_{2n+1}}$, $\mathcal{C}_0^{A B_{2n}} \simeq \mathcal{C}_0^{A D_{2n+2}}$

$\bigoplus_{\beta} R^{A^{(1)}_{2n+1}}(\beta)\text{-fmod}_0$ $\bigoplus_{\beta} R^{A D_{2n+2}}(\beta)\text{-fmod}_0$

<u>Problem</u> Construct directly the equiv. in Cor.

§5 Idea of the proof of the main thm $\mathcal{F}: \bigoplus_{\beta} R(\beta)\text{-fmod}_0 \xrightarrow{\sim} \mathcal{C}_0$

<u>Fact</u> $\exists$ block dec. $\mathcal{C}_0 = \bigoplus_{\beta} \mathcal{C}_{0,\beta}$ s.t. $\mathcal{F}_\beta: R(\beta)\text{-fmod}_0 \to \mathcal{C}_{0,\beta}$

$\therefore$ Enough to show $\mathcal{F}_\beta: R(\beta)\text{-fmod}_0 \xrightarrow{\sim} \mathcal{C}_{0,\beta}$ $(\forall \beta)$

obstruction $R(\beta)\text{-fmod}_0$ & $\mathcal{C}_{0,\beta}$ are "too small" to apply homological methods (e.g. not enough proj.)

solution find larger cat. $\widehat{R(\beta)\text{-fmod}_0}$ & $\widehat{\mathcal{C}_{0,\beta}}$ (completion)

and prove $\widehat{R(\beta)\text{-fmod}_0} \xrightarrow{\sim} \widehat{\mathcal{C}_{0,\beta}}$ instead

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$\widehat{R(\beta)}$:= the completion of $R(\beta)$ along the $\mathbb{Z}$-grading (c.f. $\mathbb{C}[z] \rightsquigarrow \mathbb{C}[[z]]$)

Consider $\widehat{R(\beta)}$-mod $\supseteq R(\beta)$-fmod$_0$ (cat. of **finitely gen'd** mod)

How to define $\widehat{\mathcal{C}_{0,\beta}}$? It is difficult to complete $U_q(\hat{\mathfrak{g}})$ since $\mathcal{C}_{0,\beta}$ is quite small!

<u>Rem</u> In the previous pf of type ADE by Fujita, $\widehat{\mathcal{C}_{0,\beta}}$ is constructed using geometrical method using quiver var.

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Set $\mathcal{E} := \mathrm{End}_{\widehat{R(\beta)}}(\hat{V}^{\otimes\beta})$, and set $\widehat{\mathscr{C}}_{0,\beta} := \mathcal{E}$-mod.

$$\begin{array}{c} \widehat{\mathscr{C}}_{0,\beta} := \mathcal{E}\text{-mod} \\ \cup\!| \\ \mathscr{C}_{0,\beta} \simeq \mathcal{E}\text{-fmod} \end{array}$$

By showing $\widehat{R(\beta)}\text{-mod} \xrightarrow{\sim} \mathcal{E}\text{-mod}$, we finally prove

$R(\beta)\text{-fmod}_0 \xrightarrow{\sim} \mathscr{C}_{0,\beta}$ ⌟

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Thank you for your concentration!