Existence of KR crystals in types $G_2^{(1)}$, $D_4^{(3)}$ and $E_6^{(1)}$

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Introduction

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(e.g. $\mathfrak{g} = \mathfrak{g}_0 \otimes \mathbb{Q}[t, t^{-1}] \oplus \mathbb{Q}K$, $\mathfrak{g}_0$: simple Lie alg.)

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Some of f.d. simple $U_q'(\mathfrak{g})$-modules have crystal bases,

but not all of them do!

Problem Classify f.d. simple $U_q'(\mathfrak{g})$-modules having crystal base.

Conjecture (Hatayama,Kuniba,Okado,Takgagi,Yamada/Tsuboi, 99-01)

Kirillov-Reshetikhin (KR) module $W^{r,\ell}$ has a crystal base.

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The conjecture holds if

(1) $W^{r,\ell}$: simple as a $U_q(\mathfrak{g}_0)$-mod. ($\mathfrak{g}_0 := \mathfrak{g}_{I\setminus\{0\}} \subseteq \mathfrak{g}$: simple subalg.)

(2) r: adjoint node (i.e. r is the neighbor of 0-node)

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   - Crystal bases and pseudobases
   - KR modules
   - Prepolarization
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crystal base and pseudobase

$U_q'(\mathfrak{g})$: quantum affine algebra,

$A := \{f(q)/g(q) \mid g(0) \neq 0\} \subseteq \mathbb{Q}(q)$: local subring,

M: integrable $U_q'(\mathfrak{g})$-module,

$e_i, f_i \curvearrowright M \quad (i \in I) \overset{\text{“twist”}}{\rightsquigarrow} \tilde{e}_i, \tilde{f}_i \curvearrowright M \ (i \in I)$: **Kashiwara operators**

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(1) A pair (L, B) is called a **crystal base** if

(a) L: A-lattice of M,

(b) $B \subseteq L/qL$: a $\mathbb{Q}$-basis,

(c) $L = \bigoplus_\lambda L_\lambda$, $B = \bigsqcup_\lambda B_\lambda$ (i.e. compatible with weight dec.),

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(e) $\tilde{e}_i b, \tilde{f}_i b \in B \sqcup \{0\}$ for $b \in B$,

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Rem. In the same way with crystal bases, from a crystal pseudobase we can construct a **crystal graph** (I-colored oriented graph)

$\rightsquigarrow$ combinatorial formulas for tensor products, branching rules, etc.

Kirillov-Reshetikhin (KR) modules

$$U'_q(\mathfrak{g}) \supseteq U_q(\mathfrak{g}_0) := \mathbb{Q}(q)\langle e_i, f_i, q^{h_i} \mid i \in I_0 := I \setminus \{0\}\rangle$$

P_0: weight lattice of $\mathfrak{g}_0$, P_0^+: set of dominant integral weights of $\mathfrak{g}_0$,

$\varpi_i \in P_0^+$ $(i \in I_0)$: fundamental weight of $\mathfrak{g}_0$ (i.e. $\langle h_i, \varpi_j\rangle = \delta_{ij}$)

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Fact {isom. classes of simple $U_q(\mathfrak{g}_0)$-modules} $\overset{1:1}{\longleftrightarrow} P_0^+$

$$\begin{array}{ccc} \uplus & & \uplus \\ V_0(\lambda) & & \lambda \end{array}$$

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$W^r_{q^k} = W^r$ as vector sp., and denoting by ρ the new action, we have

$\rho(e_i)v = q^{\delta_{0i}k} e_i v, \;\; \rho(f_i)v = q^{-\delta_{0i}k} f_i v, \;\; \rho(q^{h_i})v = q^{h_i} v.$

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For $r \in I_0$ and $\ell \in \mathbb{Z}_{>0}$, consider a nontrivial $U_q'(\mathfrak{g})$-module hom.

$$W^r_{q^{\ell-1}} \otimes W^r_{q^{\ell-3}} \otimes \cdots \otimes W^r_{q^{-\ell+1}} \xrightarrow{R} W^r_{q^{-\ell+1}} \otimes \cdots \otimes W^r_{q^{\ell-3}} \otimes W^r_{q^{\ell-1}}.$$

Definition

$W^{r,\ell} := \operatorname{Im} R$: **Kirillov-Reshetikhin** (KR) **modules**

Note $W^{r,1} = W^r$.

prepolarization

Define an anti-involution Ψ of $U_q'(\mathfrak{g})$ by

$$\Psi(e_i) = q_i^{-1} q_i^{-h_i} f_i, \quad \Psi(f_i) = q_i^{-1} q_i^{h_i} e_i, \quad \Psi(q^{h_i}) = q^{h_i},$$

where $q_i = q^{c_i}$ with a certain positive integer c_i.

Definition

Let M be a $U_q'(\mathfrak{g})$-module, and $(\ ,\ )$ a $\mathbb{Q}(q)$-bilinear form on M.

We say $(\ ,\ )$ is a **prepolarization** on M if it is symmetric

and satisfies $(xu, v) = (u, \Psi(x)v)$ for $x \in U_q'(\mathfrak{g})$ and $u, v \in M$.

Proposition

$W^{r,\ell}$ has a prepolarization $(\ ,\ )$.

Construction of this prepolarization

Recall $W^{r,\ell} := \mathrm{Im}\, R$, where

$$W^r_{q^{\ell-1}} \otimes \cdots \otimes W^r_{q^{-\ell+1}} \xrightarrow{R} W^r_{q^{-\ell+1}} \otimes \cdots \otimes W^r_{q^{\ell-1}}$$

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<u>Fact</u> W^r has a prepolarization $(\ ,\ )$.

$\rightsquigarrow$ natural pairing $(\ ,\ )$ between $W^r_{q^k}$ and $W^r_{q^{-k}}$ for any $k \in \mathbb{Z}$.

$\rightsquigarrow$ $(u_1 \otimes \cdots \otimes u_\ell, v_1 \otimes \cdots \otimes v_\ell)' = (u_1, v_1) \cdots (u_\ell, v_\ell)$ defines a pairing between $W^r_{q^{\ell-1}} \otimes \cdots \otimes W^r_{q^{-\ell+1}}$ and $W^r_{q^{-\ell+1}} \otimes \cdots \otimes W^r_{q^{\ell-1}}$.

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Then $\big(R(u), R(v)\big) := \big(u, R(v)\big)'$ for $u, v \in W^r_{q^{\ell-1}} \otimes \cdots \otimes W^r_{q^{-\ell+1}}$

Criterion for the existence of crystal pseudobase

Theorem (KKMMNN)

Let M be a f.d. $U_q'(\mathfrak{g})$-module, and assume that

(1) M has a prepolarization $(\ ,\ )$,

(2) $^{\exists}$"suitable $\mathbb{Z}$-form" $M_{\mathbb{Z}}$ in M,

(3) there exists a sequence of vectors $u_1, \ldots, u_m \in M_{\mathbb{Z}}$ s.t.

- (i) $M \cong_{U_q(\mathfrak{g}_0)} \bigoplus_{k=1}^{m} V_0\big(\mathrm{wt}(u_k)\big)$,
- (ii) $(u_k, u_j) \in \delta_{kj} + qA \quad (^{\forall}k, j)$ **(almost orthonomality)**
- (iii) $||e_i u_k||^2 \in q^{-2\langle h_i, \mathrm{wt}(u_k)\rangle - 1}A \quad (^{\forall}i \in I_0, {}^{\forall}k)$.

Then, setting

$L := \{u \in M \mid ||u||^2 \in A\}$, $B := \{b \in (M_{\mathbb{Z}} \cap L)/(M_{\mathbb{Z}} \cap qL) \mid ||b||^2 = 1\}$,

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Note (ii) $\Rightarrow b_k := \overline{u_k} \in B$, (iii) $\Rightarrow \tilde{e}_i b_k = 0 \ (i \in I_0)$.

So (i)–(iii) imply that there exist enough $U_q(\mathfrak{g}_0)$-h.w. elements in B.

what we need to do in KR module case

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(1) and (2) are known to hold for all the KR modules $W^{r,\ell}$.

Moreover, an explicit formula $W^{r,\ell} \cong_{U_q(\mathfrak{g}_0)} \bigoplus_\lambda V_0(\lambda)$ for $U_q(\mathfrak{g}_0)$-mod. decomposition has been obtained (**fermionic formula**).

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Hence what we have to do is the following:

(1) Find vectors $u_1, \dots, u_m$ s.t. $W^{r,\ell} \cong_{U_q(\mathfrak{g}_0)} \bigoplus_k V_0\big(\mathrm{wt}(u_k)\big)$,

(2) Check that these vectors satisfy (ii) and (iii).

Theorem (N)

If $\mathfrak{g}$ is of type $G_2^{(1)}$, $D_4^{(3)}$, or $E_6^{(1)}$, then the KR module $W^{r,\ell}$ has a crystal pseudobase for every r and ℓ.

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In our cases, however, this seems quite hard.

Note All the KR modules in the previous works are multiplicity-free as $U_q(\mathfrak{g}_0)$-modules, but this is not true in our cases.

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Recall We have to

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(2) check $(u_k, u_j) \in \delta_{kj} + qA$ and $||e_i u_k||^2 \in q^{-2\langle h_i, \mathrm{wt}(u_k)\rangle - 1} A$.

In the previous works [KKMMNN], [Okado-Schilling], (2) is achieved by directly calculating the values using the relations $(xu, v) = (u, \Psi(x)v)$.

In our cases, however, this seems quite hard.

Note All the KR modules in the previous works are multiplicity-free as $U_q(\mathfrak{g}_0)$-modules, but this is not true in our cases.

Hence we employ another, indirect method.

Case $\mathfrak{g} = G_2^{(1)}, D_4^{(3)}$

In both cases, 1: adjoint node

$\Rightarrow$ the existence of a crystal pseudobase for $W^{1,\ell}$ are already known.

Hence we consider $W^{2,\ell}$ only.

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Notation

$[m] = (q^m - q^{-m})/(q - q^{-1})$, $\quad [m]! = [m] \cdots [1]$,

$e_i^{(m)} = e_i^m/[m]!$ (q-devided power),

For a sequence $i_1, i_2, \dots, i_p$ of elements of I, set

$e_{i_1 i_2 \cdots i_p}^{(m)} := e_{i_1}^{(m)} e_{i_2}^{(m)} \cdots e_{i_p}^{(m)}$.

$v_\ell \in W^{r,\ell}$: a vector with weight $\ell\varpi_2$ s.t. $||v_\ell||^2 = 1$.

(here we regard $\ell\varpi_2 \in P_0$ as a weight of $\mathfrak{g}$ by setting $\langle K, \ell\varpi_2 \rangle = 0$)

vectors $\{u_k\}$ in the criterion

Proposition

Define the set of vectors S in $W^{2,\ell}$ by

$$\{e_0^{(a)}e_1^{(b)}e_2^{(c)}e_{10}^{(d)}v_\ell \mid 3b \le c \le b+d, a \le b, -c+3d \le \ell\} \quad (\mathfrak{g}\colon\ G_2^{(1)})$$

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Then we have $W^{2,\ell} \cong_{U_q(\mathfrak{g}_0)} \bigoplus_{u\in S} V_0\big(\mathrm{wt}(u)\big)$, and we have

$(u,v) \in \delta_{uv} + qA$ and $||e_i u||^2 \in q^{-2\langle h_i, \mathrm{wt}(u)\rangle - 1}A$ for $u, v \in S$.

How to find the set S?

Observation In all the cases of previous works, the vectors of $W^{r,\ell}$ in the criterion are written in the form $e_{i_1}^{(a_1)} \cdots e_{i_p}^{(a_p)} v_\ell$ $(a_1, \dots, a_p \in \mathbb{Z}_{\geq 0})$.
Here $s_{i_1} \cdots s_{i_p}$: a reduced expression of an element w in the affine Weyl group, which satisfies that $w(\varpi_r + \Lambda_0)$ is a dominant integral weight of $\mathfrak{g}$ (Λ_0: fund. weight of $\mathfrak{g}$).

By assuming this also holds in our cases,
the set of vectors $S = \{e_0^{(a)} e_1^{(b)} e_2^{(c)} e_{10}^{(d)} v_\ell \mid \cdots\}$ was found.

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Rem. At L/qL, $\overline{e_0^{(a)} e_1^{(b)} e_2^{(c)} e_{10}^{(d)} v_\ell} \neq \widetilde{e}_0{}^a \widetilde{e}_1{}^b \widetilde{e}_2{}^c \widetilde{e}_1{}^d \widetilde{e}_0{}^d v_\ell$ in general.

Proof for $G_2^{(1)}$, $D_4^{(3)}$

Recall $W^{r,\ell} := \operatorname{Im} R$, where

$$W^r_{q^{\ell-1}} \otimes \cdots \otimes W^r_{q^{-\ell+3}} \otimes W^r_{q^{-\ell+1}} \xrightarrow{R} W^r_{q^{-\ell+1}} \otimes \cdots \otimes W^r_{q^{\ell-3}} \otimes W^r_{q^{\ell-1}},$$

and $(R(u), R(v)) = (u, R(v))'$.

Lemma

$$W_q^{r,\ell-1} \otimes W^{r,1}_{q^{-\ell+1}} \overset{\exists R_1}{\twoheadrightarrow} W^{r,\ell} \overset{\exists R_2}{\hookrightarrow} W^{r,\ell-1}_{q^{-1}} \otimes W^{r,1}_{q^{\ell-1}},$$

and $(R_1(u), R_1(v)) = (u, R_2 \circ R_1(v))''$ for $u, v \in W_q^{r,\ell-1} \otimes W^{r,1}_{q^{1-\ell}}$,

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$= (\sum x^{(1)} v_{\ell-1} \otimes x^{(2)} v_1, \sum y^{(1)} v_{\ell-1} \otimes y^{(2)} v_1)''$

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$= (\sum x^{(1)} v_{\ell-1} \otimes x^{(2)} v_1, \sum y^{(1)} v_{\ell-1} \otimes y^{(2)} v_1)''$

$= \sum (x^{(1)} v_{\ell-1}, y^{(1)} v_{\ell-1})(x^{(2)} v_1, y^{(2)} v_1)$. We can use the induction on ℓ.

in type $E_6^{(1)}$: vectors in the criterion

$r = 1, 2, 6$: already known, $r = 3, 5$: proved by a direct calculation.

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Proposition

Define the set of vectors S by

$\{e_0^{(a)} e_1^{(b)} e_2^{(c)} e_{432}^{(d)} e_{65432}^{(e)} e_{10}^{(f)} v_\ell \mid a \le b \le c \le d \le e, c+d+e \le b+f, f \le c+\ell\}$

Then we have $W^{4,\ell} \cong_{U_q(\mathfrak{g}_0)} \bigoplus_{u \in S} V_0\big(\mathrm{wt}(u)\big)$, and we have

$(u, v) \in \delta_{uv} + qA$, and $||e_i u||^2 \in q^{-2\langle h_i, \mathrm{wt}(u)\rangle - 1} A$ for $u, v \in S$.

It is difficult to apply the method used in the case $G_2^{(1)}, D_4^{(3)}$!

The main reason: $W^{4,1}$ is already complicated.

$\dim W^{2,1} = 8$ $(G_2^{(1)})$, $\dim W^{2,1} = 31$ $(D_4^{(3)})$, $\dim W^{4,1} = 3732$ $(E_6^{(1)})$.

extremal weight modules

affine weight $\mu \in P \rightsquigarrow$ extremal weight module $V(\mu)$ [Kashiwara, 94]
($U_q(\mathfrak{g})$-mod. with a generator u_μ of weight μ and certain defining rel.)

Note μ: positive (resp. negative) level $\rightsquigarrow V(\mu)$: h.w (resp. l.w) mod.
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Theorem (Kashiwara, 94)

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Theorem (Beck-Nakajima, 04)

$V(\mu)$ has a prepolarization $(\ ,\ )$, and we have $(G(b), G(b')) \in \delta_{bb'} + qA$.

For $\boldsymbol{a} = (a_1, \ldots, a_6) \in \mathbb{Z}_{\geq 0}^{\times 6}$, $e^{\boldsymbol{a}} := e_0^{(a_1)} e_1^{(a_2)} e_2^{(a_3)} e_{432}^{(a_4)} e_{65432}^{(a_5)} e_{10}^{(a_6)}$.

We will give a sketch of the proof for $(e^{\boldsymbol{a}} v_\ell, e^{\boldsymbol{a'}} v_\ell) \in \delta_{\boldsymbol{aa'}} + qA$.

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$(e^{\boldsymbol{a}} v_\ell, e^{\boldsymbol{a}'} v_\ell) = 0$ if $\boldsymbol{a} \neq \boldsymbol{a}'$.

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<u>Fact</u> $^\exists$hom. $V(\ell\Lambda_4) \otimes V(-3\ell\Lambda_0) \twoheadrightarrow V(\ell\varpi_4)$ preserving global bases.

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<u>Cor.</u> $||e^{\boldsymbol{a}} u_{\ell\varpi_4}||^2 \in 1 + qA$ by the previous theorem.

Lemma

$||e^{\boldsymbol{a}} u_{\ell\varpi_4}||^2$ (in $V(\ell\varpi_4)$) $= ||e^{\boldsymbol{a}}(v_1)^{\otimes\ell}||^2$ (in $\left(W^{4,1}\right)^{\otimes\ell}$).

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(ii) $V(\varpi_4) \cong \mathbb{Q}[z, z^{-1}] \otimes W^{4,1} \Rightarrow {}^\exists$hom. $V(\varpi_4) \overset{(z=1)}{\twoheadrightarrow} W^{4,1}$.

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Check $V(\ell\varpi_4) \hookrightarrow V(\varpi_4)^{\otimes\ell} \overset{(z=1)^{\otimes\ell}}{\twoheadrightarrow} \left(W^{4,1}\right)^{\otimes\ell}$ preserves $||e^{\boldsymbol{a}} * ||^2$. □

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By combining this with the previous corollary, we have $||e^{\boldsymbol{a}}(v_1)^{\otimes\ell}||^2 \in 1 + qA$.

Now it suffices to show the following:

Lemma

$||e^{\boldsymbol{a}}(v_1)^{\otimes \ell}||^2$ (in $(W^{4,1})^{\otimes \ell}$) $= ||e^{\boldsymbol{a}} v_\ell||^2$ (in $W^{4,\ell}$).

For simplicity, assume $\ell = 2$.

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$= \sum_{\boldsymbol{b}} q^{2c(\boldsymbol{b})} ||e^{\boldsymbol{a}-\boldsymbol{b}} v_1||^2 ||e^{\boldsymbol{b}} v_1||^2$ $(\because (e^{\boldsymbol{b}} v_1, e^{\boldsymbol{b}'} v_1) = 0$ unless $\boldsymbol{b} = \boldsymbol{b}')$

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$||e^{\boldsymbol{a}}(v_1)^{\otimes 2}||^2 = ||\sum_{\boldsymbol{b}} q^{c(\boldsymbol{b})} e^{\boldsymbol{a}-\boldsymbol{b}} v_1 \otimes e^{\boldsymbol{b}} v_1||^2$

$= \sum_{\boldsymbol{b},\boldsymbol{b}'} q^{c(\boldsymbol{b})+c(\boldsymbol{b}')}(e^{\boldsymbol{a}-\boldsymbol{b}}v_1, e^{\boldsymbol{a}-\boldsymbol{b}'}v_1)(e^{\boldsymbol{b}}v_1, e^{\boldsymbol{b}'}b_1)$

$= \sum_{\boldsymbol{b}} q^{2c(\boldsymbol{b})}||e^{\boldsymbol{a}-\boldsymbol{b}}v_1||^2||e^{\boldsymbol{b}}v_1||^2$ $(\because (e^{\boldsymbol{b}}v_1, e^{\boldsymbol{b}'}v_1) = 0$ unless $\boldsymbol{b} = \boldsymbol{b}')$

$||e^{\boldsymbol{a}}v_2||^2 = (e^{\boldsymbol{a}}(v_1)^{\otimes 2}, e^{\boldsymbol{a}}(v_1)^{\otimes 2})'$ (on $(W_q^4 \otimes W_{q^{-1}}^4) \times (W_{q^{-1}}^4 \otimes W_q^4)$)

Now it suffices to show the following:

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$$= \sum_{\boldsymbol{b},\boldsymbol{b'}} q^{c(\boldsymbol{b})+c(\boldsymbol{b'})} (e^{\boldsymbol{a}-\boldsymbol{b}} v_1, e^{\boldsymbol{a}-\boldsymbol{b'}} v_1)(e^{\boldsymbol{b}} v_1, e^{\boldsymbol{b'}} b_1)$$
$$= \sum_{\boldsymbol{b}} q^{2c(\boldsymbol{b})} ||e^{\boldsymbol{a}-\boldsymbol{b}} v_1||^2 ||e^{\boldsymbol{b}} v_1||^2 \quad (\because (e^{\boldsymbol{b}} v_1, e^{\boldsymbol{b'}} v_1) = 0 \text{ unless } \boldsymbol{b} = \boldsymbol{b'})$$

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$$= \left(\sum_{\boldsymbol{b}} q^{c(\boldsymbol{b})+d(\boldsymbol{b})} e^{\boldsymbol{a}-\boldsymbol{b}} v_1 \otimes e^{\boldsymbol{b}} v_1, \sum_{\boldsymbol{b}} q^{c(\boldsymbol{b})-d(\boldsymbol{b})} e^{\boldsymbol{a}-\boldsymbol{b}} v_1 \otimes e^{\boldsymbol{b}} v_1\right)$$

Now it suffices to show the following:

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$= \sum_{\boldsymbol{b}} q^{2c(\boldsymbol{b})} ||e^{\boldsymbol{a}-\boldsymbol{b}} v_1||^2 ||e^{\boldsymbol{b}} v_1||^2.$ □

Comments on other types

The remaining types: $E_7^{(1)}$, $E_8^{(1)}$, $F_4^{(1)}$, $E_6^{(2)}$.

In these types, since the fermionic formula is quite complicated,

we cannot find so far a candidate of the set of vectors in the criterion.

Fact fermionic formula $= \#\{$**rigged configuration**$\}$

Can we construct a bijection between vectors in the criterion and rigged configurations?