SOME ESTIMATES OF THE MORSE-NOVIKOV NUMBERS FOR KNOTS AND LINKS

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We present some results on estimates of the Morse-Novikov numbers for knots and links. Using these, we show the Morse-Novikov numbers concretely for some knots and links.

Keywords: Morse-Novikov number, Circle valued Morse map, knot, link

1. Introduction

These notes are adapted from the talk given at the conference 'Intelligence of Low Dimensional Topology 2006' at Hiroshima University.

We define a circle valued Morse map for knots and links as follows, and then we argue some methods to estimate the number of critical points. We present methods using Heegaard splitting and so on. Note that this Morse map may be regarded as a generalization of Milnor map ([8]). It is studied from this viewpoint and some methods to estimate are given recently. See [6] and [12]. Further, there are several works studying a gradient flow corresponding to this Morse map. See, for example, [4] and [9].

Let L be an oriented link in the 3-sphere S^3. A Morse map $f : C_L := S^3 - L \to S^1$ is said to be *regular* if each component of L, say L_i, has a neighborhood framed as $S^1 \times D^2$ such that (i) $L_i = S^1 \times \{0\}$ (ii) $f|_{S^1\times(D^2-\{0\})} \to S^1$ is given by $(x, y) \to y/|y|$. We denote by $m_i(f)$ the number of the critical points of f of index i. A Morse map $f : C_L \to S^1$ is said to be *minimal* if for each i the number $m_i(f)$ is minimal on the class of all regular maps homotopic to f.

Under these notations, the following basic theorem is shown ([10]).

Theorem 1.1 ([10]). *There is a minimal Morse map satisfying:*
(1) $m_0(f) = m_3(f) = 0$*;*

(2) *All critical values of the same index coincide;*
(3) $f^{-1}(x)$ *is a Seifert surface of* L *for any regular value* x.

The minimal Morse map satisfying the conditions in Theorem 1.1 is said to be *moderate*. We define the Morse-Novikov number of L as follows:

$$\mathcal{MN}(L) = \min\left\{\sum_i m_i(f) \mid f : C_L \to S^1 \text{ is a regular Morse map}\right\}.$$

Then we can observe the following. See [10] for the detail.

Proposition 1.1. (1) $\mathcal{MN}(L) = 0$ *if and only if* L *is fibred.*
(2) $\mathcal{MN}(L) = 2 \times \min\{m_1(f) \mid f$ *is moderate*$\}$.

2. Heegaard splitting for sutured manifolds and product decompositions

We recall the definition of a sutured manifold ([1]). A *sutured manifold* (M, γ) is a compact oriented 3-dimensional manifold M together with a set $\gamma(\subset \partial M)$ of mutually disjoint annuli $A(\gamma)$ and tori $T(\gamma)$. In this paper, we deal with the case of $T(\gamma) = \emptyset$. The core curve of a component of $A(\gamma)$ is called a *suture*, and we denote by $s(\gamma)$ the set of sutures. Every component of $R(\gamma) = \partial M - \text{Int}A(\gamma)$ is oriented, and $R_+(\gamma)(R_-(\gamma)$ resp.) denotes the union of the components whose normal vectors point out of (into resp.) M. Moreover, the orientations of $R(\gamma)$ must be coherent with respect to the orientations of $s(\gamma)$. Let (V, γ) be a sutured manifold such that V is a 3-ball and γ is an annulus embedded in ∂V. Then we call (V, γ) the *trivial sutured manifold*. In this case $R_+(\gamma)$ is a disk and $R_-(\gamma)$ is also a disk.

We say that a sutured manifold (M, γ) is a *product sutured manifold* if (M, γ) is homeomophic to $(F \times [0, 1], \partial F \times [0, 1])$ with $R_+(\gamma) = F \times \{1\}, R_-(\gamma) = F \times \{0\}, A(\gamma) = \partial F \times [0, 1]$, where F is a compact surface. Let L be an oriented link in S^3 and R a Seifert surface for L. The *exterior* $E(L)$ of L is the closure of $S^3 - N(L; S^3)$. Then $R \cap E(L)$ is homeomorphic to R, and we often abbreviate $R \cap E(L)$ to R. $(N, \delta) = (N(R; E(L)), N(\partial R; \partial E(L))$ has a product sutured manifold structure $(R \times [0, 1], \partial R \times [0, 1])$. So (N, δ) is called the *product sutured manifold for* R. We say that the sutured manifold $(N^c, \delta^c) = (E(L) - \text{Int}N, \partial E(L) - \text{Int}\delta))$ with $R_\pm(\delta^c) = R_\mp(\delta)$ is the *complementary sutured manifold for* R. A Seifert surface R is a fiber surface if and only if the complementary sutured manifold for R is a product sutured manifold. Note that we say that an oriented surface R in S^3 is a *fiber surface* if ∂R is a fibred link with R a fiber.

A *product disk* $D(\subset M)$ is a properly embedded disk such that ∂D intersects $s(\gamma)$ transversely in two points. We obtain a new sutured manifold (M', γ') from (M, γ) by cutting M along D and extending $s(\gamma) - \mathrm{Int}N(D)$ in the natural way (see Fig. 1). This decomposition $(M, \gamma) \xrightarrow{D} (M', \gamma')$ is called a *product decomposition*.

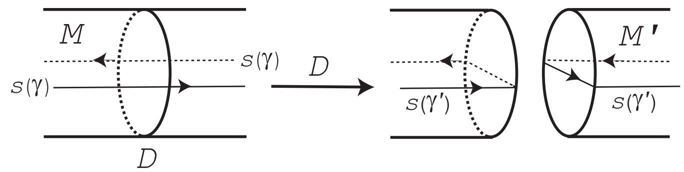

Fig. 1. Product decomposition

In [1], the next theorem is proved:

Theorem 2.1 ([1]). *A sutured manifold* (M, γ) *is a product sutured manifold if and only if there exists a sequence of product decompositions:*

$$(M, \gamma) \xrightarrow{D_1} (M_1, \gamma_1) \xrightarrow{D_2} \cdots \xrightarrow{D_n} (M_n, \gamma_n)$$

such that (M_n, γ_n) *is a union of the trivial sutured manifolds.*

A *compression body* W is a cobordism rel ∂ between surfaces $\partial_+ W$ and $\partial_- W$ such that W is homeomorphic to $\partial_+ W \times [0, 1] \cup$ (2-handles) $\cup$ (3-handles) and $\partial_- W$ has no 2-sphere components. It is easy to see that if $\partial_- W \neq \emptyset$ and W is connected, W is obtained from $\partial_- W \times [0, 1]$ by attaching a number of 1-handles along the disks on $\partial_- W \times \{1\}$ where $\partial_- W$ corresponds to $\partial_- W \times \{0\}$. We denote by $h(W)$ the number of these attaching 1-handles.

Let (M, γ) be a sutured manifold such that $R_+(\gamma) \cup R_-(\gamma)$ has no 2-sphere components. We say that (W, W') is a *Heegaard splitting* of (M, γ) if both W and W' are compression bodies, $M = W \cup W'$ with $W \cap W' = \partial_+ W = \partial_+ W', \partial_- W = R_+(\gamma)$, and $\partial_- W' = R_-(\gamma)$. Assume that $R_+(\gamma)$ is homeomorphic to $R_-(\gamma)$. Then we define the *handle number* $h(M, \gamma)$ of (M, γ) as follows:

$$h(M, \gamma) = \min\{h(W)(= h(W')) \mid (W, W') \text{ is a Heegaard splitting of } (M, \gamma)\}.$$

If (M, γ) is the complementary sutured manifold for a Seifert surface R, we may define the handle number of R as follows:

$$h(R) = \min\{h(W) \mid (W, W') \text{ is a Heegaard splitting of } (M, \gamma)\}.$$

Note that $h(R) = 0$ if and only if R is a fiber surface. In this setting, we can have the next proposition. The detail can be seen in [5] and [10].

Proposition 2.1. *Let L be an oriented link in S^3. Then*

$$\mathcal{MN}(L) = 2 \times \min\{h(R) \mid R \text{ is a Seifert surface of } L\}.$$

We show some examples in the next section. The next lemma is a way to estimate $h(R)$ and $\mathcal{MN}(L)$, cf. Proposition 5.2 in [6].

Lemma 2.1. *Let R be a Seifert surface of an oriented link L in S^3 and (M, γ) the complementary sutured manifold for R. If there exist a set of arcs $\{\alpha_i\}$ $(i = 1, 2, \ldots, 2n)$ properly embedded in M such that $\partial\alpha_i \subset R_+(\gamma)$ for $i = 1, \ldots, n$, $\partial\alpha_i \subset R_-(\gamma)$ for $i = n+1, \ldots, 2n$ and the sutured manifold $(M - \cup_{i=1}^{2n} \mathrm{Int}N(\alpha_i), \gamma)$ is a product sutured manifold. Then, $h(R) \leq n$.*

Proof. We may regard $N(R_+(\gamma) \cup (\cup_{i=1}^{n} \alpha_i))$ ($N(R_-(\gamma) \cup (\cup_{i=n+1}^{2n} \alpha_i))$ resp.) as a compression body W (W' resp.) such that $h(W) = n$ ($h(W') = n$ resp.). Since $(M - \cup_{i=1}^{2n} \mathrm{Int}N(\alpha_i), \gamma)$ is a product sutured manifold, $M - (\mathrm{Int}W \cup \mathrm{Int}W')$ is homeomorphic to $\partial_+W \times [0, 1]$ such that $\partial_+W = \partial_+W \times \{0\}$ and $\partial_+W' = \partial_+W \times \{1\}$. Set $\overline{W} = W \cup (\partial_+W \times [0, 1/2])$ and $\overline{W}' = W' \cup (\partial_+W \times [1/2, 1])$. Then $(\overline{W}, \overline{W}')$ is a Heegaard splitting for (M, γ) with $h(\overline{W}) = h(\overline{W}') = n$. This completes the proof of this lemma. □

3. The Morse-Novikov numbers for prime knots of ≤ 10 crossings and links of ≤ 9 crossings

In this section, we assume that a knot or link is prime. The fibred knots up to 10 crossings and fibred links up to 9 crossings have been detected by Kanenobu ([7]) and Gabai ([1]). Note that there exist 2^{n-1} orientation classes to analyze for a given link of n components. By using Heegaard splitting associated to the Morse map, we can show that $\mathcal{MN}(L) = 2$ if L is a non-fibred knot up to 10 crossings or link up to 9 crossings. The knot case has been argued in [3]. The followings are some examples. We use Rolfsen's notation ([11]) to describe a given unoriented link.

Example 3.1. Let L be the oriented trivial link with 2-components, and we may regard two component disks R as a Seifert surface of L. We denote by (M, γ) the complementary sutured manifold of R. Then M is homeomorphic to $S^2 \times S^1$, $\gamma (= A(\gamma))$ consists of two annuli, and $R_\pm(\gamma)$ is two component disks. See Fig. 2. Let α_1 (α_2 resp.) be an arc properly embedded in M as illustrated in Fig. 2, and set $M' = M - \mathrm{Int}N(\alpha_1 \cup \alpha_2)$. Then we may regarded (M', γ) as a sutured manifold such that M' is homeomorphic to $D^2 \times S^1$. By the product decomposition as in Fig. 3 and Theorem 2.1, we have (M', γ) is a product sutured manifold. Thus we obtain that $\mathcal{MN}(L) \leq 2 \times h(R) \leq 2$ by Lemma 2.1. Since it is known that L is not fibred, we have $\mathcal{MN}(L) = 2 \times h(R) = 2$.

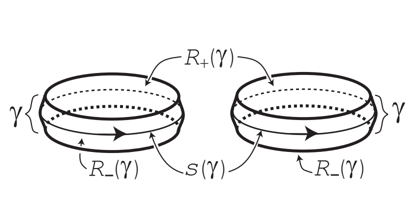

Fig. 2.

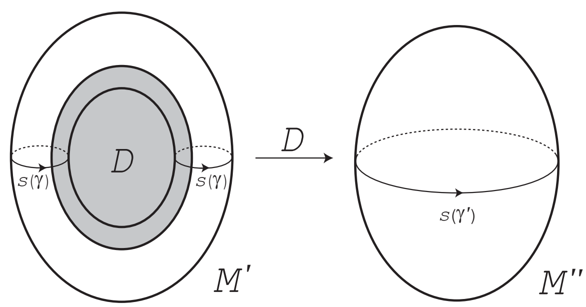

Fig. 3.

Example 3.2. Let L be 9^2_5 with the orientation as illustrated in Fig. 4. The oriented links in Fig. 4 are the same links. Let R be a Seifert surface as in Fig. 5 and α_1 and α_2 arcs properly embedded in the complementary sutured manifold for R. Then, by the same argument as in Example 3.1, we have $\mathcal{MN}(L) = 2 \times h(R) = 2$. Note that L has a Seifert surface $\widetilde{R}$ such that $h(\widetilde{R}) = 2$.

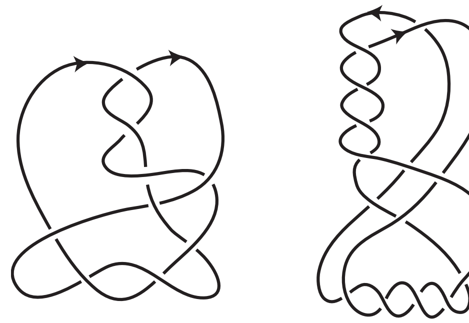

Fig. 4.

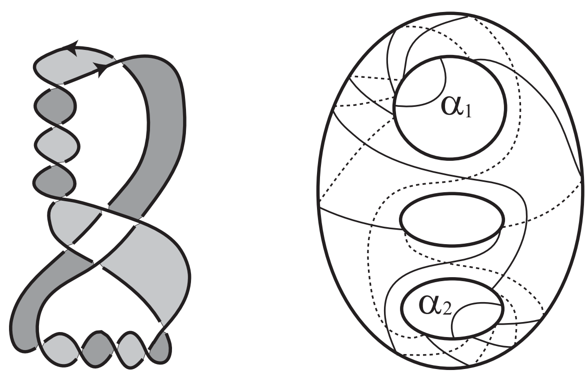

Fig. 5.

4. Connected sum

The behavior of the handle number under a Murasugi sum has been studied in [2] and [3]. From the result, we have:

Theorem 4.1. *Let $K_1 \sharp K_2$ be a connected sum of K_1 and K_2. Then,* $\mathcal{MN}(K_1 \sharp K_2) \leq \mathcal{MN}(K_1) + \mathcal{MN}(K_2)$.

However the study of the behavior of the Morse-Novikov number under a connected sum is not complete, that is, the next natural question is still open as far as I know.

Question 4.1. $\mathcal{MN}(K_1 \sharp K_2) = \mathcal{MN}(K_1) + \mathcal{MN}(K_2)$?

The behavior of the Morse-Novikov number under a Murasugi sum is studied. For the detail, see Section 7 in [6].

Theorem 4.2 ([6]). *For any $m \in \mathbb{Z}$, there exist knots K_1 and K_2 such that $\mathcal{MN}(K_1 * K_2) \geq \mathcal{MN}(K_1) + \mathcal{MN}(K_2) + m$, where $*$ is Murasugi (not connected) sum.*

5. Novikov homology and the Alexander invariant

There is an estimate of the Morse-Novikov number using the Alexander invariant. Let $\Lambda = \mathbb{Z}[t, t^{-1}]$ and $\widehat{\Lambda} = \mathbb{Z}[[t]][t^{-1}]$. We usually use Λ to discuss about the knot theory concerning the infinite cyclic covering of a knot complement. However, the ring $\widehat{\Lambda}$ is convenient here. Note that $\widehat{\Lambda}$ is a principal ideal domain.

Set $\widehat{H}_i(L) = H_i(\overline{C}_L) \otimes_\Lambda \widehat{\Lambda}$ and $\widehat{b}_i(L) = \operatorname{rank}_{\widehat{\Lambda}} \widehat{H}_i(L)$, where $\overline{C}_L$ is the (usual) infinite cyclic covering of an oriented link L. The homology $\widehat{H}_i(L)$ is called the *Novikov homology*. Let $\widehat{q}_i(L)$ be the minimal number of $\widehat{\Lambda}$-generators of the torsion submodule of $\widehat{H}_i(L)$.

By an analog of the ordinary Morse theory and homological arguments, we can have the following estimates.

Theorem 5.1 ([10]).

$$m_i(f) \geq \widehat{b}_i(L) + \widehat{q}_i(L) + \widehat{q}_{i-1}(L).$$

Corollary 5.1.

$$\mathcal{MN}(L) \geq 2(\widehat{b}_1(L) + \widehat{q}_1(L)).$$

Since $\widehat{\Lambda}$ is a principal ideal domain, the $\widehat{H}_*(L)$ can be decomposed into cyclic modules, which relates the Alexander invariants. In fact, we have:

Theorem 5.2 ([10]). $\widehat{H}_1(L) = \oplus_{s=0}^{m-1} \widehat{\Lambda}/\gamma_s \widehat{\Lambda}$, *where $\gamma_s = \Delta_s/\Delta_{s+1}$ and Δ_s is the s-th Alexander polynomial. Thus,*
(1) $\widehat{b}_1(L)$ *is equal to the number of the polynomials Δ_s that are equal to 0.*
(2) $\widehat{q}_1(L)$ *is equal to the number of the γ_s that are nonzero and nonmonic.*

There are some concrete examples to calculate the Morse-Novikov numbers using these results in [4]. The next question is proposed in [6]. We denote by $g(K)$ the genus of a knot K.

Question 5.1 ([6]). *Does there exist a knot K with $\mathcal{MN}(K) > 2g(K)$?*

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